

Bisimulation Equivalence of First-Order Grammars^{*}

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Abstract. A decidability proof for bisimulation equivalence of first-order grammars (finite sets of labelled rules for rewriting roots of first-order terms) is presented. The equivalence generalizes the DPDA (deterministic pushdown automata) equivalence, and the result corresponds to the result achieved by Sénizergues (1998, 2005) in the framework of equational graphs, or of PDA with restricted ε -steps. The framework of classical first-order terms seems particularly useful for providing a proof that should be understandable for a wider audience. We also discuss an extension to branching bisimilarity, announced by Fu and Yin (2014).

1 Introduction

Decision problems for semantic equivalences have been a frequent topic in computer science. E.g., for pushdown automata (PDA) *language equivalence* was quickly shown undecidable, while the decidability in the case of deterministic PDA (DPDA) is a famous result by Sénizergues [15]. A finer equivalence, called *bisimulation equivalence* or *bisimilarity*, has emerged as another fundamental behavioural equivalence; for deterministic systems it essentially coincides with language equivalence. We name [1] to exemplify the first decidability results for infinite-state systems, and refer to [18] for a survey of a specific area.

One of the most involved results in the area [17] shows the decidability of bisimilarity of equational graphs with finite out-degree (or of PDA with deterministic popping ε -steps); this generalizes the result for DPDA. The recent nonelementary lower bound [2] for the problem is, in fact, TOWER-hardness in the terminology of [14], and it holds even for real-time PDA, i.e. PDA with no ε -steps. For the full above mentioned PDA the problem is even not primitive recursive, since it is Ackermann-hard [12]. In the deterministic case, the equivalence problem is known to be PTIME-hard, and has a primitive recursive upper bound shown by Stirling [20]; a finer analysis places the problem in TOWER [12]. This complexity gap is just one indication that the respective fundamental equivalence problems are far from being fully understood. Another such indication might be the length and the technical nature of the so far published proofs (including the unpublished [21]).

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This paper is an attempt to make a further step in clarifying the main decidability proof in the mentioned area. It provides a self-contained decidability proof for bisimulation equivalence in labelled transition systems generated by *first-order grammars* (FO-grammars), which seems to be a particularly convenient formalism. The states are here first-order terms over a specified finite set of function symbols (or “nonterminals”); the transitions are induced by a finite set of labelled rules that allow to rewrite the roots of terms. This framework is equivalent to the framework of [17]; cf., e.g., [5] for the early references, or [10] for a concrete transformation of PDA to FO-grammars (which is also given in Appendix here). The proof in this paper is in principle based on the same high-level ideas as the proof in [17] but it is shorter and simpler; we do not provide a detailed comparison here. This paper is also a (self-contained) continuation of [10] where the first-order term framework was used to give a decidability proof in the deterministic case.

Related work is also discussed in Section 4, where we address the extension of decidability to branching bisimilarity, studied recently by Y. Fu and Q. Yin [7].

Remark. Some parts are formatted as this remark; they contain additional details and comments. The aim of this paper is to make the proof easily understandable, not technically shortest.

2 Preliminaries and Result

In this section we define the basic notions and state the result. Some standard definitions are restricted when we do not need the full generality.

By \mathbb{N} we denote the set $\{0, 1, 2, \dots\}$ of nonnegative integers; we use $[i, j]$ to denote the set $\{i, i+1, \dots, j\}$. For a set \mathcal{A} , by \mathcal{A}^* we denote the set of finite sequences of elements of \mathcal{A} , which are also called *words* (over \mathcal{A}). By $|w|$ we denote the *length* of $w \in \mathcal{A}^*$. By ε we denote the *empty sequence* (hence $|\varepsilon| = 0$).

LTSs. A *labelled transition system* (an LTS) is a tuple

$$\mathcal{L} = (\mathcal{S}, \Sigma, (\xrightarrow{a})_{a \in \Sigma})$$

where \mathcal{S} is a *finite or countable* set of *states*, Σ is a finite set of *actions* (or *letters*), and $\xrightarrow{a} \subseteq \mathcal{S} \times \mathcal{S}$ is a set of *a-transitions* (for each $a \in \Sigma$). In fact, we only deal with image-finite LTSs, where $\mathcal{L} = (\mathcal{S}, \Sigma, (\xrightarrow{a})_{a \in \Sigma})$ is *image-finite* if the set $\{s' \mid s \xrightarrow{a} s'\}$ is finite for each pair $s \in \mathcal{S}$, $a \in \Sigma$. We say that \mathcal{L} is a *deterministic LTS* if for each pair $s \in \mathcal{S}$, $a \in \Sigma$ there is at most one s' such that $s \xrightarrow{a} s'$.

By $s \xrightarrow{w} s'$, where $w = a_1 a_2 \dots a_n \in \Sigma^*$, we denote that there is a *path* $s = s_0 \xrightarrow{a_1} s_1 \xrightarrow{a_2} \dots \xrightarrow{a_n} s_n = s'$; if $s \xrightarrow{w} s'$, then s' is *reachable from* s , within $|w|$ steps. By $s \xrightarrow{w}$ we denote that w is *enabled by* s , i.e., $s \xrightarrow{w} s'$ for some s' . A *state* s is *dead* if there is no $a \in \Sigma$ such that $s \xrightarrow{a}$.

If \mathcal{L} is deterministic, then by $s \xrightarrow{w} s'$ or $s \xrightarrow{w}$ we also denote the respective unique path.

(Stratified) bisimilarity. Let $\mathcal{L} = (\mathcal{S}, \Sigma, (\xrightarrow{a})_{a \in \Sigma})$ be a given LTS. We say that a set $\mathcal{B} \subseteq \mathcal{S} \times \mathcal{S}$ *covers* $(s, t) \in \mathcal{S} \times \mathcal{S}$ if

- for any $a \in \Sigma$ and $s' \in \mathcal{S}$ such that $s \xrightarrow{a} s'$ there is $t' \in \mathcal{S}$ such that $t \xrightarrow{a} t'$ and $(s', t') \in \mathcal{B}$, and
- for any $a \in \Sigma$ and $t' \in \mathcal{S}$ such that $t \xrightarrow{a} t'$ there is $s' \in \mathcal{S}$ such that $s \xrightarrow{a} s'$ and $(s', t') \in \mathcal{B}$.

We note that if s, t are dead states, then (s, t) is covered by any $\mathcal{B} \subseteq \mathcal{S} \times \mathcal{S}$, in particular by \emptyset . If there is an action $a \in \Sigma$ that is enabled by precisely one of s, t , then (s, t) is not covered by any \mathcal{B} .

For $\mathcal{B}, \mathcal{B}' \subseteq \mathcal{S} \times \mathcal{S}$ we say that \mathcal{B}' *covers* \mathcal{B} if \mathcal{B}' covers each $(s, t) \in \mathcal{B}$. A set $\mathcal{B} \subseteq \mathcal{S} \times \mathcal{S}$ is a *bisimulation* if \mathcal{B} covers \mathcal{B} . States $s, t \in \mathcal{S}$ are *bisimilar*, written $s \sim t$, if there is a bisimulation \mathcal{B} containing (s, t) . We note the standard fact that $\sim \subseteq \mathcal{S} \times \mathcal{S}$ is the maximal bisimulation, the union of all bisimulations.

We put $\sim_0 = \mathcal{S} \times \mathcal{S}$. For $k \in \mathbb{N}$, $\sim_{k+1} \subseteq \mathcal{S} \times \mathcal{S}$ is the set of all pairs covered by \sim_k . We easily verify that \sim and \sim_k are equivalence relations, and that $\sim_0 \supseteq \sim_1 \supseteq \sim_2 \supseteq \dots \supseteq \sim$. For the (first infinite) ordinal ω we put $s \sim_\omega t$ if $s \sim_k t$ for all $k \in \mathbb{N}$; hence $\sim_\omega = \bigcap_{k \in \mathbb{N}} \sim_k$. It is a standard fact that $\bigcap_{k \in \mathbb{N}} \sim_k$ is a bisimulation in any image-finite LTS, where we thus have $\sim = \sim_\omega$.

Eq-levels. Given an image-finite LTS, we attach the *equivalence level* (eq-level) to each pair of states:

$$\text{EqLv}(s, t) = \max \{k \in \mathbb{N} \cup \{\omega\} \mid s \sim_k t\}.$$

First-order-term LTSs informally. We focus on certain (image-finite) LTSs in which states are first-order terms; we mean standard finite terms primarily but it will turn out convenient to consider also infinite regular terms (i.e. infinite terms with only finitely many pairwise different subterms). The terms are built from *variables* from a fixed countable set

$$\text{VAR} = \{x_1, x_2, x_3, \dots\}$$

and from *function symbols*, also called (*ranked*) *nonterminals*, from some specified finite set \mathcal{N} ; each $A \in \mathcal{N}$ has *arity*(A) $\in \mathbb{N}$. We use A, B, C, D for nonterminals, while E, F, \dots (possibly with subscripts etc.) are reserved for terms. An example of a (standard finite) term is $E_1 = A(D(x_5, C(x_2, B)), x_5, B)$ where the arities of A, B, C, D are 3, 0, 2, 2, respectively. The left-hand side of Fig. 1 depicts the syntactic tree of E_1 . (The right-hand side E_2 will be referred to later.)

Transitions are determined by a finite set of *root-rewriting* rules. An example of a “non-popping” rule is $A(x_1, x_2, x_3) \xrightarrow{a} C(D(x_3, B), x_2)$, an example of a “popping” rule is $A(x_1, x_2, x_3) \xrightarrow{b} x_1$. Each rule induces the transitions arising by applying the same substitution σ to both the left-hand side (lhs) and the right-hand side (rhs) of the rule. E.g., the rule

$$A(x_1, x_2, x_3) \xrightarrow{a} C(D(x_3, B), x_2) \text{ and the substitution } \sigma \text{ for which } \\ \sigma(x_1) = D(x_5, C(x_2, B)), \sigma(x_2) = x_5, \sigma(x_3) = B$$

(where $A(x_1, x_2, x_3)$ after applying σ becomes $A(D(x_5, C(x_2, B)), x_5, B)$) induce the transition $A(D(x_5, C(x_2, B)), x_5, B) \xrightarrow{a} C(D(B, B), x_5)$ depicted in Fig. 2. Fig. 2 depicts an a -transition between two states, where the states in our LTSs

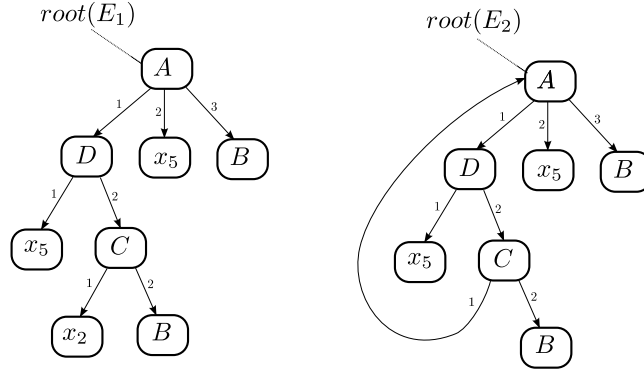


Fig. 1. Syntactic tree of $E_1 = A(D(x_5, C(x_2, B)), x_5, B)$, and a graph presenting E_2

are terms. The small symbols x_1, x_2, x_3 are superfluous here, they just depict the original variables in the lhs and in the rhs of the respective rule; these variables have been replaced by applying the substitution σ . Hence x_3 and x_2 in the target-term have been replaced by the third root-successor and by the second root-successor of the source-term, respectively. In this concrete case the first root-successor of the source-term “disappears” since x_1 does not occur in the rhs of the rule.

Another example can be given by the rule $A(x_1, x_2, x_3) \xrightarrow{b} x_1$ and the above σ , which induces the transition, or the one-step path, $F \xrightarrow{b} H$ where $F = A(D(x_5, C(x_2, B)), x_5, B)$ and $H = D(x_5, C(x_2, B))$. In this case our one-step path exposes a root-successor H in the source term F (the first root-successor in our case); the path has thus “sunked” to a subterm in depth 1.

The result informally. We will show that there is an algorithm that computes $\text{EQLV}(T_0, U_0)$ when given a finite set of root-rewriting rules and two terms T_0, U_0 . In the rest of this section we formalize this statement, making also some conventions about our use of (finite and infinite) terms and substitutions.

Regular terms, presentation size. We identify terms with their syntactic trees, and denote them by E, F, \dots . Thus a *term* E over \mathcal{N} (where \mathcal{N} is a set of ranked nonterminals) is a rooted, ordered, finite or infinite tree where each node has a label from $\mathcal{N} \cup \text{VAR}$; if the label of a node is $x_i \in \text{VAR}$, then the node has no successors, and if the label is $A \in \mathcal{N}$, then it has m (immediate) successor-nodes where $m = \text{arity}(A)$. More precisely, a term corresponds to a set of isomorphic trees, since two isomorphic trees represent the same term. Each node is also the root of a *subterm* of E , i.e., of the subtree rooted in this node; more precisely, each concrete node is the root of a *subterm-occurrence*, since a subterm corresponds to a set of isomorphic subtrees. A subterm can thus have more (maybe infinitely many) occurrences in E . Each *subterm-occurrence* has its (nesting) *depth* in E , which is its (naturally defined) distance from the root of E ; the term E itself is a subterm-occurrence with depth 0. E.g., in $E_1 = A(D(x_5, C(x_2, B)), x_5, B)$

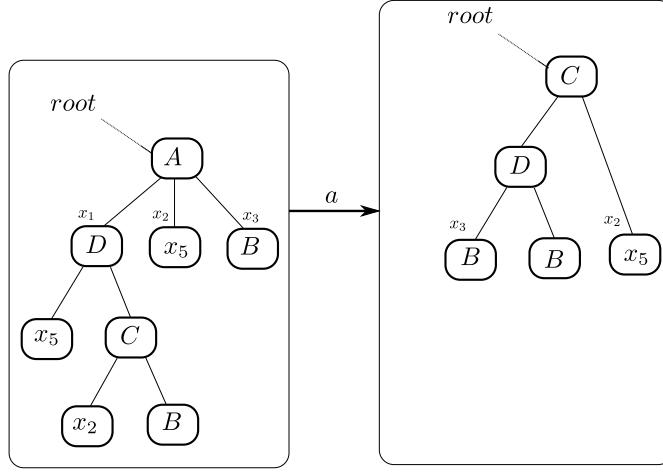


Fig. 2. Transition generated by $A(x_1, x_2, x_3) \xrightarrow{a} C(D(x_3, B), x_2)$ and σ (in the text)

(in Fig. 1) there is one occurrence of the term $C(x_2, B)$, with depth 2, and two occurrences of x_5 , with depths 1 and 2.

We also use the standard notation: a term is either x_i or $A(G_1, \dots, G_m)$; if $E = A(G_1, \dots, G_m)$, then $\text{ROOT}(E) = A \in \mathcal{N}$, $m = \text{arity}(A)$, and G_1, \dots, G_m are the *root-successors*, i.e., the ordered subterm-occurrences with depth 1.

A *term* E is *finite* if the respective tree is finite; by $\text{HEIGHT}(E)$ we then mean the largest depth of a subterm-occurrence in E . For $E_1 = A(D(x_5, C(x_2, B)), x_5, B)$ in Fig. 1 we thus have $\text{HEIGHT}(E_1) = 3$.

A (possibly infinite) *term* is *regular* if it has only finitely many subterms (though the subterms may be infinite and can have infinitely many occurrences). Any regular term has a natural *finite-graph presentation* (with possible cycles). E.g., the right-hand side of Fig. 1 presents a regular term E_2 ; here the term E_2 itself is a subterm with infinitely many occurrences (with depths 0, 3, 6, ...). By $\text{PRSIZE}(E)$ (the presentation size of E) we mean the size of the smallest graph presentation of E .

We can be more precise, though the respective notions are standard. A *finite-graph presentation* of a (regular) term over \mathcal{N} is a finite directed (multi)graph, with a designated root, where each node has a label from $\mathcal{N} \cup \text{VAR}$; if the label of a node is $x_i \in \text{VAR}$, then the node has no outgoing arcs, and if the label is $A \in \mathcal{N}$, then the node has $\text{arity}(A)$ ordered outgoing arcs. The standard “tree-unfolding” of the graph is the respective term, which is infinite if there are cycles in the graph. We can obviously effectively compare if two graph presentations represent the same term. Given a presentation of a regular term E , we can thus compute the *syntactic graph* of E , i.e., the graph whose nodes (one-to-one) correspond to the (roots of) subterms occurring in E . (E.g., the syntactic graph of E_2 in Fig. 1 arises from the given graph presentation by

merging the nodes with label x_5 and merging those with label B ; but we note that if we replaced D with C , we *could not* merge the nodes with label C .) We can take the number of nodes of the syntactic graph of E as $\text{PRSIZE}(E)$.

In what follows, by a “term” we mean a “regular term” if we do not say explicitly that the term is finite. (We do not consider non-regular terms.) We reserve symbols E, F, G, H , and also T, U, V, W , for denoting (regular) terms.

Substitutions, associative composition. By $\text{TERMS}_{\mathcal{N}}$ we denote the set of all (regular) terms over a set \mathcal{N} of (ranked) nonterminals. A *substitution* σ is a mapping

$$\sigma : \text{VAR} \rightarrow \text{TERMS}_{\mathcal{N}} \text{ whose support } \text{SUPP}(\sigma) = \{x_i \mid \sigma(x_i) \neq x_i\}$$

is *finite*; we reserve the symbol σ for substitutions. By $\text{RANGE}(\sigma)$ we mean the set $\{\sigma(x_i) \mid x_i \in \text{SUPP}(\sigma)\}$. By *applying a substitution* σ to a term E we get the term $E\sigma$ that arises from E by replacing each occurrence of x_i with $\sigma(x_i)$. Hence $E = x_i$ implies $E\sigma = x_i\sigma = \sigma(x_i)$; we *prefer the notation* $x_i\sigma$ to $\sigma(x_i)$.

The *composition of substitutions*, where $\sigma = \sigma_1\sigma_2$ satisfies $x_i\sigma = (x_i\sigma_1)\sigma_2$, can be easily verified to be associative. We thus write simply $E\sigma_1\sigma_2$ when meaning $(E\sigma_1)\sigma_2$ or $E(\sigma_1\sigma_2)$.

First-order grammars. A *first-order grammar*, an *FO-grammar* or just a *grammar* for short, is a tuple $\mathcal{G} = (\mathcal{N}, \Sigma, \mathcal{R})$ where \mathcal{N} is a finite set of ranked *nonterminals*, viewed as function symbols with arities, Σ is a finite set of *actions* (or letters), and \mathcal{R} is a finite set of *rules* of the form

$$A(x_1, x_2, \dots, x_m) \xrightarrow{a} E$$

where $A \in \mathcal{N}$, $\text{arity}(A) = m$, $a \in \Sigma$, and E is a *finite* term over \mathcal{N} in which each occurring variable is from the set $\{x_1, x_2, \dots, x_m\}$.

Rule-based and action-based LTSs generated by grammars. Given $\mathcal{G} = (\mathcal{N}, \Sigma, \mathcal{R})$, by $\mathcal{L}_{\mathcal{G}}^{\mathcal{R}} = (\text{TERMS}_{\mathcal{N}}, \mathcal{R}, (\xrightarrow{r})_{r \in \mathcal{R}})$ we denote the (*rule based*) LTS where each rule r of the form $A(x_1, x_2, \dots, x_m) \xrightarrow{a} E$ induces $(A(x_1, \dots, x_m))\sigma \xrightarrow{r} E\sigma$ for any substitution σ . (Hence also $A(x_1, \dots, x_m) \xrightarrow{r} E$, due to σ with $\text{SUPP}(\sigma) = \emptyset$.)

Speaking in an informal “operational” manner, we can apply a rule r of the form $A(x_1, \dots, x_m) \xrightarrow{r} E$ to (a graph-presentation of) F iff $\text{ROOT}(F) = A$. If so, and $E = x_j$, then the target of the j -th outgoing arc of the root of F (which might be the root itself in the case of a loop) is the root of H where $F \xrightarrow{r} H$. If $E \notin \text{VAR}$, we get H (for which $F \xrightarrow{r} H$) by adding (a fresh copy of) E to F where the root of E becomes the root of the arising H ; to finish the construction of H , each arc in E leading to a node labelled with x_j is redirected to the j -th root-successor in F . Hence the variables x_1, \dots, x_m in the rules serve just as “place-holders” for root-successors (in the source term of a transition); recall again Fig. 2.

The LTS $\mathcal{L}_{\mathcal{G}}^{\mathcal{R}}$ is deterministic, since for each F and r there is at most one H such that $F \xrightarrow{r} H$. Hence $F \xrightarrow{w}$, for $w \in \mathcal{R}^*$, refers to a unique path in $\mathcal{L}_{\mathcal{G}}^{\mathcal{R}}$ (which is later technically convenient).

We stress explicitly that *transitions cannot add variables*, i.e., $F \xrightarrow{w} H$ implies that each variable occurring in H also occurs in F (though not vice versa in general; recall that the first root-successor “disappeared” by the transition in Fig. 2, which also caused that the subterm x_2 “disappeared”). We also note that $F \xrightarrow{w} H$ implies $F\sigma \xrightarrow{w} H\sigma$ for any substitution σ ; this follows from the fact that x_i are dead, not enabling any action.

Finally we observe that our stipulation that the right-hand sides (rhs) E in the grammar-rules $A(x_1, \dots, x_m) \xrightarrow{a} E$ are finite implies that *all terms reachable from a finite term are finite*. (It turns out technically convenient to have the rhs finite while including regular terms into our LTSs.)

By the *action-based* LTS, related to a grammar $\mathcal{G} = (\mathcal{N}, \Sigma, \mathcal{R})$, we mean the LTS $\mathcal{L}_{\mathcal{G}}^A = (\text{TERMS}_{\mathcal{N}}, \Sigma, (\xrightarrow{a})_{a \in \Sigma})$ where each rule $A(x_1, \dots, x_m) \xrightarrow{a} E$ induces

$$(A(x_1, \dots, x_m))\sigma \xrightarrow{a} E\sigma$$

for any substitution σ .

Hence $F \xrightarrow{w} H$ in $\mathcal{L}_{\mathcal{G}}^R$ implies $F \xrightarrow{\text{ACT}(w)} H$ in $\mathcal{L}_{\mathcal{G}}^A$, where $\text{ACT}(w)$ is the naturally defined *action-image* of w : the homomorphism $\text{ACT} : \mathcal{R}^* \rightarrow \Sigma^*$ is defined by putting $\text{ACT}(r) = a$ for any rule r of the form $A(x_1, \dots, x_m) \xrightarrow{a} E$.

We note that $\mathcal{L}_{\mathcal{G}}^A$ is image-finite, and nondeterministic in general. In fact, we still *complete the definition of $\mathcal{L}_{\mathcal{G}}^A$ by stipulating that*

$$\text{no } \mathcal{B} \subseteq \text{TERMS}_{\mathcal{N}} \times \text{TERMS}_{\mathcal{N}} \text{ covers } (x_i, H) \text{ or } (H, x_i) \text{ when } H \neq x_i.$$

We thus have that

$$x_i \neq H \text{ implies } x_i \not\sim_1 H, \text{ i.e., } \text{EQLV}(x_i, H) = 0.$$

In particular we have $x_i \not\sim_1 x_j$ for $i \neq j$. Technically we think of each used variable $x \in \text{VAR}$ as being equipped with its unique action a_x and with the transition $x \xrightarrow{a_x} x$ in $\mathcal{L}_{\mathcal{G}}^A$; this entails that $x_i \not\sim_1 H$ for $H \neq x_i$ without any special stipulation.

Convention. Whenever we consider $F \xrightarrow{w} H$ in $\mathcal{L}_{\mathcal{G}}^A$, we tacitly assume that no special transitions $x \xrightarrow{a_x} x$ are involved. Hence $F \xrightarrow{w} H$ implies $F\sigma \xrightarrow{w} H\sigma$ for any substitution σ . We also stipulate that \emptyset covers (x_i, x_i) , thus avoiding superfluous technicalities.

The stipulation $x_i \not\sim_1 H$ for $H \neq x_i$ reflects the fact that $x_i \neq H$ implies that $x_i\sigma \not\sim_1 H\sigma$ for some σ , unless the underlying grammar \mathcal{G} is trivial. The special transitions $x \xrightarrow{a_x} x$ are just one technical possibility how to reflect this fact in $\mathcal{L}_{\mathcal{G}}^A$ smoothly.

In what follows we refer to the action-based LTSs $\mathcal{L}_{\mathcal{G}}^A$, if we do not say explicitly that we have $\mathcal{L}_{\mathcal{G}}^R$ in mind.

Theorem 1. *There is an algorithm that, given an FO-grammar $\mathcal{G} = (\mathcal{N}, \Sigma, \mathcal{R})$ and $T_0, U_0 \in \text{TERMS}(\mathcal{N})$, computes $\text{EQLV}(T_0, U_0)$ in $\mathcal{L}_{\mathcal{G}}^A$.*

3 Proof of Theorem 1

We note that deciding \sim_0 is trivial, since $T \sim_0 U$ holds for all T, U . When having a procedure deciding \sim_k , we can easily construct a procedure deciding \sim_{k+1} ; this follows from the facts that $T \sim_{k+1} U$ iff (T, U) is covered by \sim_k , and that for any V we can construct all (finitely many) pairs (a, V') such that $V \xrightarrow{a} V'$. We thus get a part of Theorem 1:

Proposition 2. *There is an algorithm that, given \mathcal{G} and T_0, U_0 , outputs $\text{EqLv}(T_0, U_0)$ if $T_0 \not\sim U_0$, and does not halt if $T_0 \sim U_0$.*

We need to modify the algorithm so that it recognizes the case $T_0 \sim U_0$ in finite time. As a convenient tool we introduce a round-based game between Prover(she) and Refuter(he); the game is more involved than the standard bisimulation game. We start with a simple first version of the game, and then we enhance it stepwise. Refuter will be always able to force his win in finite time if the terms in the initial pair (T_0, U_0) are non-equivalent. Prover will be always able to avoid losing if $T_0 \sim U_0$, but only in the last game-version she will be able to force her win in finite time. Since Prover's winning strategy for $T_0 \sim U_0$ in the last game-version will be finitely presentable and effectively verifiable, a proof of Theorem 1 will be finished. Before the first game-version we observe some simple standard facts related to (stratified) bisimulation equivalence.

Expansions. Assume an LTS $\mathcal{L} = (\mathcal{S}, \Sigma, (\xrightarrow{a})_{a \in \Sigma})$. By

$$\mathcal{B} \triangleleft \mathcal{B}', \text{ where } \mathcal{B}, \mathcal{B}' \subseteq \mathcal{S} \times \mathcal{S},$$

we denote that \mathcal{B}' is a *minimal expansion for \mathcal{B}* , i.e., \mathcal{B}' covers \mathcal{B} and no proper subset of \mathcal{B}' covers \mathcal{B} ; this also implies that for each $(s', t') \in \mathcal{B}'$ there is $(s, t) \in \mathcal{B}$ such that $s \xrightarrow{a} s'$ and $t \xrightarrow{a} t'$ for some $a \in \Sigma$. We note that $\emptyset \triangleleft \emptyset$, and if s, t are dead (not enabling any action), then $\{(s, t)\} \triangleleft \emptyset$.

For any $k \in \mathbb{N}$ we have $k < \omega$ and we stipulate $\omega - k = \omega + k = \omega$. We also stipulate $\min \emptyset = \omega$, and define $\text{MINEQL}(\mathcal{B}) = \min\{\text{EqLv}(s, t) \mid (s, t) \in \mathcal{B}\}$.

Proposition 3.

- (1) If $\text{MINEQL}(\mathcal{B}) = 0$ then there is no \mathcal{B}' such that $\mathcal{B} \triangleleft \mathcal{B}'$.
- (2) If $\mathcal{B} \triangleleft \mathcal{B}'$ and $\text{MINEQL}(\mathcal{B}) < \omega$, then $\text{MINEQL}(\mathcal{B}) > \text{MINEQL}(\mathcal{B}')$.
- (3) If $\text{MINEQL}(\mathcal{B}) > 0$ then there is \mathcal{B}' such that $\mathcal{B} \triangleleft \mathcal{B}'$ and $\text{MINEQL}(\mathcal{B}') \geq \text{MINEQL}(\mathcal{B}) - 1$. (In particular, if $\mathcal{B} \subseteq \sim$ then $\mathcal{B} \triangleleft \mathcal{B}'$ for some $\mathcal{B}' \subseteq \sim$.)
- (4) For $k \in \mathbb{N}$ we have $s \sim_k t$ iff there is a sequence $\{(s, t)\} \triangleleft \mathcal{B}_1 \triangleleft \mathcal{B}_2 \triangleleft \dots \triangleleft \mathcal{B}_k$.

Prover-Refuter game (first version). A play starts with a grammar $\mathcal{G} = (\mathcal{N}, \Sigma, \mathcal{R})$ and an *initial pair* (T_0, U_0) of terms. For $i = 0, 1, 2, \dots$, the $(i+1)$ -th round of the play starts with some specified pair (T_i, U_i) and proceeds as follows:

1. Prover chooses $k > 0$ and some $\mathcal{B}_j \subseteq \text{TERMS}_{\mathcal{N}} \times \text{TERMS}_{\mathcal{N}}$ for $j = 1, 2, \dots, k$ and shows that $\mathcal{B}_0 \triangleleft \mathcal{B}_1 \triangleleft \mathcal{B}_2 \triangleleft \dots \triangleleft \mathcal{B}_k$ where $\mathcal{B}_0 = \{(T_i, U_i)\}$.
If this is impossible (i.e., if $T_i \not\sim_1 U_i$), then Refuter wins.

2. Refuter chooses a pair (T'_i, U'_i) in $\mathcal{B}_k \setminus \bigcup_{j=0}^{k-1} \mathcal{B}_j$. If this is impossible, i.e. if $\mathcal{B}_k \subseteq \bigcup_{j=0}^{k-1} \mathcal{B}_j$ (which includes the case $\mathcal{B}_k = \emptyset$), then Prover wins. (In this case $T_i \sim U_i$, since $T_i \not\sim U_i$ implies that $\mathcal{B}_k \not\subseteq \bigcup_{j=0}^{k-1} \mathcal{B}_j$, by Prop. 3(2).)
3. The pair $(T_{i+1}, U_{i+1}) = (T'_i, U'_i)$ is taken for starting the $(i+2)$ -th round.

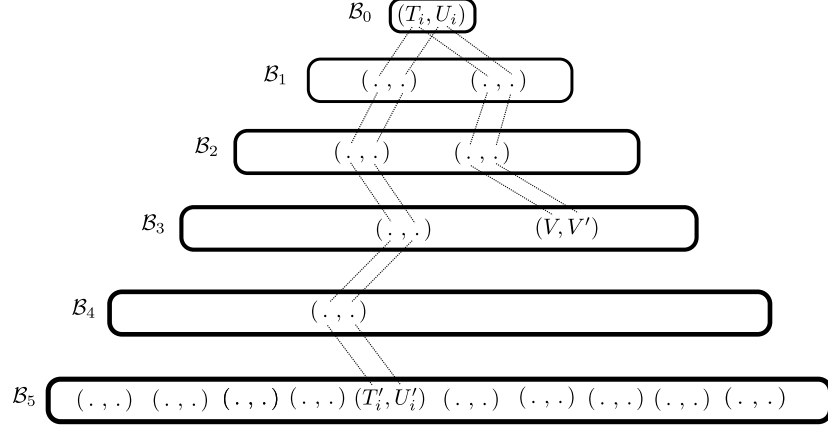


Fig. 3. Illustration of a game-round (where Prover has chosen $k = 5$)

Fig. 3 illustrates an $(i+1)$ -th round (with $k=5$). Note that our requirement $\mathcal{B}_0 \triangleleft \mathcal{B}_1 \triangleleft \dots \triangleleft \mathcal{B}_k$ entails that there are $u_1, u_2 \in \mathcal{R}^*$ such that $|u_1| = |u_2| = k$, $\text{ACT}(u_1) = \text{ACT}(u_2)$, and $T_i \xrightarrow{u_1} T'_i, U_i \xrightarrow{u_2} U'_i$ (in $\mathcal{L}_{\mathcal{G}}^R$), as sketched in the figure.

The reversals of u_1, u_2 can be found by the “bottom-up approach”, starting from (T'_i, U'_i) and going up to (T_i, U_i) . This follows from recalling that $\mathcal{B} \triangleleft \mathcal{B}'$ implies that for each $(G', H') \in \mathcal{B}'$ there is $(G, H) \in \mathcal{B}$ such that $G \xrightarrow{a} G'$ and $H \xrightarrow{a} H'$ for some $a \in \Sigma$.

We also note that, e.g., for each V such that $T_i \xrightarrow{v} V$ in $\mathcal{L}_{\mathcal{G}}^R$ where $|v| = j \leq k$ there is $v' \in \mathcal{R}^*$ and V' such that $|v'| = |v|$, $\text{ACT}(v) = \text{ACT}(v')$, $U_i \xrightarrow{v'} V'$, and $(V, V') \in \mathcal{B}_j$.

This is also depicted in Fig. 3. When looking for $U_i \xrightarrow{v'} V'$, we now use the “top-down approach” driven by $T_i \xrightarrow{v} V$.

We say that *Refuter* uses the *least-eqlevel strategy*, if he always chooses (T'_i, U'_i) so that $\text{EqLv}(T'_i, U'_i) = \text{MINEqL}(\mathcal{B}_k)$; in this case $\text{EqLv}(T'_i, U'_i) < \text{MINEqL}(\bigcup_{j=0}^{k-1} \mathcal{B}_j)$, and thus $\text{EqLv}(T'_i, U'_i) < \text{EqLv}(T_i, U_i) - (k-1)$, unless $T \sim U$ for all $(T, U) \in \bigcup_{j=0}^k \mathcal{B}_j$. We easily observe the following facts.

Proposition 4. Let $\text{EqLv}(T_0, U_0) = e \in \mathbb{N} \cup \{\omega\}$.

1. If $e < \omega$, then *Refuter* wins within $e+1$ rounds by the *least-eqlevel strategy*.
2. *Prover* can guarantee that she will not lose within e rounds.

Prover's additional tool. A challenge is to add sound possibilities for Prover to enable her to force her win in finite time if $T_0 \sim U_0$. We allow Prover to claim a win when she can (soundly) demonstrate, in some $(i+1)$ -th round, that either Refuter has not used the least-eqlevel strategy or $T_0 \sim U_0$. This new abstract rule does not change Prop. 4. A simple instance is a *repeat*: if $\{T_i, U_i\} = \{T_j, U_j\}$ for some $j < i$, then Prover can claim her win. (Equality $T_i = U_i$ is another trivial example.)

We thus further assume that Prover wins when a repeat appears, and we look at more involved options enabling her to “balance”, i.e., to replace T'_i, U'_i (in the point 3 of a game-round) with T_{i+1}, U_{i+1} that are “closer” to each other, while keeping $\text{EqLV}(T_{i+1}, U_{i+1}) = \text{EqLV}(T'_i, U'_i)$ when Refuter uses the least eq-level strategy. Before formulating the second version of the game, we clarify the crucial underlying facts. First a trivial one:

Proposition 5. *Assume an LTS \mathcal{L} . If $\text{EqLV}(s, t) = k$ and $\text{EqLV}(s, s') > k$, then $\text{EqLV}(s', t) = k$ (since $s' \sim_k s \sim_k t$ and $s' \sim_{k+1} s \not\sim_{k+1} t$).*

Congruence, the crux of balancing. We assume a given grammar $\mathcal{G} = (\mathcal{N}, \Sigma, \mathcal{R})$. For substitutions σ, σ' and $k \in \mathbb{N} \cup \{\omega\}$ we put

$$\sigma \sim_k \sigma' \text{ if } x_i \sigma \sim_k x_i \sigma' \text{ for each } x_i \in \text{VAR}.$$

We also put $\text{EqLV}(\sigma, \sigma') = \max \{k \in \mathbb{N} \cup \{\omega\} \mid \sigma \sim_k \sigma'\}$; hence $\text{EqLV}(\sigma, \sigma') = \min \{\text{EqLV}(x_i \sigma, x_i \sigma') \mid x_i \in \text{VAR}\}$. We now note that \sim_k and $\sim = \sim_\omega$ are congruences:

Proposition 6.

- (1) *If $E \sim_k F$, then $E\sigma \sim_k F\sigma$; hence $\text{EqLV}(E, F) \leq \text{EqLV}(E\sigma, F\sigma)$.*
 - (2) *If $\sigma \sim_k \sigma'$, then $E\sigma \sim_k E\sigma'$; hence $\text{EqLV}(\sigma, \sigma') \leq \text{EqLV}(E\sigma, E\sigma')$.*
- Moreover, if $\sigma \sim_k \sigma'$ and $E \notin \text{VAR}$ (i.e., $\text{ROOT}(E) \in \mathcal{N}$), then $E\sigma \sim_{k+1} E\sigma'$.*

Proof. (1) Suppose $\{(E, F)\} = \mathcal{B}_0 \triangleleft \mathcal{B}_1 \triangleleft \mathcal{B}_2 \cdots \triangleleft \mathcal{B}_k$; note that any pair (x_i, H) , or (H, x_i) , in $\bigcup_{j=0}^{k-1} \mathcal{B}_j$ must satisfy $H = x_i$ (since otherwise it cannot be covered by our definition of \mathcal{L}_G^Δ). For each $j \in [0, k]$ we put $\mathcal{B}'_j = \{(G\sigma, H\sigma) \mid (G, H) \in \mathcal{B}_j\}$. We almost get $\{(E\sigma, F\sigma)\} = \mathcal{B}'_0 \triangleleft \mathcal{B}'_1 \triangleleft \mathcal{B}'_2 \triangleleft \cdots \triangleleft \mathcal{B}'_k$; just for the cases $(x_i, x_i) \in \mathcal{B}_j$, $j \in [0, k-1]$, where $(x_i \sigma, x_i \sigma)$ is not covered by \mathcal{B}'_{j+1} , we complete $\mathcal{B}'_{j+1}, \mathcal{B}'_{j+2}, \dots, \mathcal{B}'_k$ with some pairs of identical terms, using the fact that $\{(E, E)\} \triangleleft \{(E', E') \mid E \xrightarrow{a} E' \text{ for some } a \in \Sigma\}$.

(2) Suppose $\sigma \sim_k \sigma'$, and take $\{(E, E)\} = \mathcal{B}_0 \triangleleft \mathcal{B}_1 \triangleleft \mathcal{B}_2 \cdots \triangleleft \mathcal{B}_k$ where $\bigcup_{j=0}^k \mathcal{B}_j \subseteq \{(F, F) \mid F \in \text{TERMS}_{\mathcal{N}}\}$. Let $\mathcal{B}'_j = \{(G\sigma, G\sigma') \mid (G, G) \in \mathcal{B}_j\}$. For the cases $(x_i, x_i) \in \mathcal{B}_j$, and thus $(x_i \sigma, x_i \sigma') \in \mathcal{B}'_j$, we complete $\mathcal{B}'_{j+1}, \mathcal{B}'_{j+2}, \dots, \mathcal{B}'_k$ accordingly, using the fact that $x_i \sigma \sim_k x_i \sigma'$. If $E \notin \text{VAR}$, then this procedure is valid even when we start with $\{(E, E)\} = \mathcal{B}_0 \triangleleft \mathcal{B}_1 \triangleleft \mathcal{B}_2 \cdots \triangleleft \mathcal{B}_{k+1}$. \square

Remark. The compositionality induced by the congruence properties leads naturally to considering the following modification of our game, based on decompositions. Prover is always allowed to “decompose” (T_i, U_i) , i.e., to present some finite set \mathcal{B} of pairs of terms that are in some sense smaller than (T_i, U_i) ,

if it is guaranteed that $\text{EqLv}(T_i, U_i) \geq \text{MinEqL}(\mathcal{B})$; Refuter then chooses a pair from \mathcal{B} for continuing. If our measure of size satisfies that there are only finitely many pairs with the size that is smaller than or equal to the size of any given (T, U) , then Refuter's least-eqllevel strategy still wins if the initial terms are non-equivalent. On the other hand, if there is a bound such that each pair $T \sim U$ that is bigger than the bound is decomposable via a set $\mathcal{B} \subseteq \sim$, then we have a required algorithm: if Prover keeps decomposing large pairs via subsets of \sim , then we get a repeat eventually.

This is a basis of decision algorithms for so called BPA processes, which can be viewed as being generated by FO-grammars where all nonterminals have arity 1 (or 0). We can refer, e.g., to the papers [4,3,11] for details. However, in our more general case such a straightforward approach does not seem clear. We follow a more involved way, based on a balancing strategy that makes the component-terms in (T_i, U_i) “close to each other”. We note that balancing strategies, in different frameworks, were used by Sénizergues [15,16,17] and then Stirling [19,20,21]. In fact, we will also discuss a bit of decomposition later, in Section 4.

We now illustrate how Prover can use already the simple fact captured by Prop. 5. Suppose the $(i+1)$ -th round starts with (T_i, U_i) and Refuter chooses (T'_i, U'_i) in \mathcal{B}_k (we refer to the notation in the game definition, and to Fig. 3). We thus have $T_i \xrightarrow{u_1} T'_i, U_i \xrightarrow{u_2} U'_i$ in \mathcal{L}_G^R , for some $u_1, u_2 \in \mathcal{R}^*$, where $|u_1| = |u_2| = k$ (and $\text{ACT}(u_1) = \text{ACT}(u_2)$).

Suppose that $T_i \xrightarrow{u_1} T'_i$ is *not* a shortest path from T_i to T'_i (in \mathcal{L}_G^R); then we have $T_i \xrightarrow{v_1} T'_i$ for some $v_1 \in \mathcal{R}^*$ where $|v_1| < |u_1|$. As we already observed, we then must also have $U_i \xrightarrow{v_2} U''$ for some U'' and some $v_2 \in \mathcal{R}^*$ such that $|v_1| = |v_2|$ and $(T'_i, U'') \in \bigcup_{j=0}^{k-1} \mathcal{B}_j$. (In Fig. 3 we would have a respective pair $(V, V') = (T'_i, U'')$.) We thus have $\text{EqLv}(T'_i, U'') > \text{EqLv}(T'_i, U'_i)$ when $T'_i \not\sim U'_i$ and Refuter uses the least-eqllevel strategy.

Therefore Refuter “cannot protest” when Prover puts $(T_{i+1}, U_{i+1}) = (U'', U'_i)$ instead of $(T_{i+1}, U_{i+1}) = (T'_i, U'_i)$, since $\text{EqLv}(T'_i, U'_i) = \text{EqLv}(U'', U'_i)$ if Refuter uses the least-eqllevel strategy. We note that U'', U'_i are close to each other in the sense that they are both reachable within k steps from one “pivot term”, namely U_i . We have $U_i \rightsquigarrow_k (U'', U'_i)$, where generally we define

$$\begin{aligned} W \rightsquigarrow_k T &\Leftrightarrow_{\text{df}} W \xrightarrow{v} T \text{ for some } v \text{ of length at most } k, \text{ and} \\ W \rightsquigarrow_k (T, U) &\Leftrightarrow_{\text{df}} W \rightsquigarrow_k T \text{ and } W \rightsquigarrow_k U. \end{aligned} \tag{1}$$

In the second game-version below we use the congruence properties to enable Prover to replace (T'_i, U'_i) with “closer” (T_{i+1}, U_{i+1}) even in some cases where $T_i \xrightarrow{u_1} T'_i$ is a shortest path from T_i to T'_i and $U_i \xrightarrow{u_2} U'_i$ is a shortest path from U_i to U'_i .

Prover-Refuter game (second version). The only change w.r.t. the first game-version is in the point 3:

3. Prover creates (T_{i+1}, U_{i+1}) for the start of the $(i+2)$ -th round:
 Either she puts $(T_{i+1}, U_{i+1}) = (T'_i, U'_i)$, thus making *no change*, or she can use one of the following options if available:

- i/ *Left-balancing*: Prover presents T'_i as $G\sigma$ for some *finite term* G and some substitution σ , where for each $V \in \text{RANGE}(\sigma)$ she finds V' such that $(V, V') \in \bigcup_{j=0}^{k-1} \mathcal{B}_j$. She defines σ' with $\text{SUPP}(\sigma') = \text{SUPP}(\sigma)$ as follows: if $\sigma(x_\ell) = V$, then $\sigma'(x_\ell) = V'$, where (V, V') is an above found pair. Finally she puts $(T_{i+1}, U_{i+1}) = (G\sigma', U'_i)$.
- ii/ *Right-balancing*: Symmetrically, Prover presents U'_i as $G\sigma$, finds all appropriate pairs (V', V) in $\bigcup_{j=0}^{k-1} \mathcal{B}_j$, and puts $(T_{i+1}, U_{i+1}) = (T'_i, G\sigma')$.

Our previous illustration, where $T_i \xrightarrow{u_i} T'_i$ was not a shortest path from T_i to T'_i , was a special case: we had $T'_i = G\sigma$ where $G = x_1$, $\text{SUPP}(\sigma) = \{x_1\}$ and $x_1\sigma = T'_i$, and we replaced $G\sigma$ with $G\sigma'$ where $x_1\sigma' = U''$ (and thus $(x_i\sigma, x_i\sigma') \in \bigcup_{j=0}^{k-1} \mathcal{B}_j$ for all $x_i \in \text{SUPP}(\sigma) = \text{SUPP}(\sigma') = \{x_1\}$).

Informally speaking, in the second game-version Prover might replace the whole T'_i with some U'' that is “shortly reachable from the pivot” (if possible), but she can also replace just “small-depth” subterms V of T'_i with (sub)terms V' that are “shortly reachable from the pivot”; in the latter case some “*special finite head*” G of T'_i remains. (The case of right-balancings is symmetric.)

A left-balancing (with the pivot $W = U_i$ and the bal-result $(G\sigma', U'_i)$) is also depicted in the upper part of Fig. 4. (The lower part will be discussed later.)

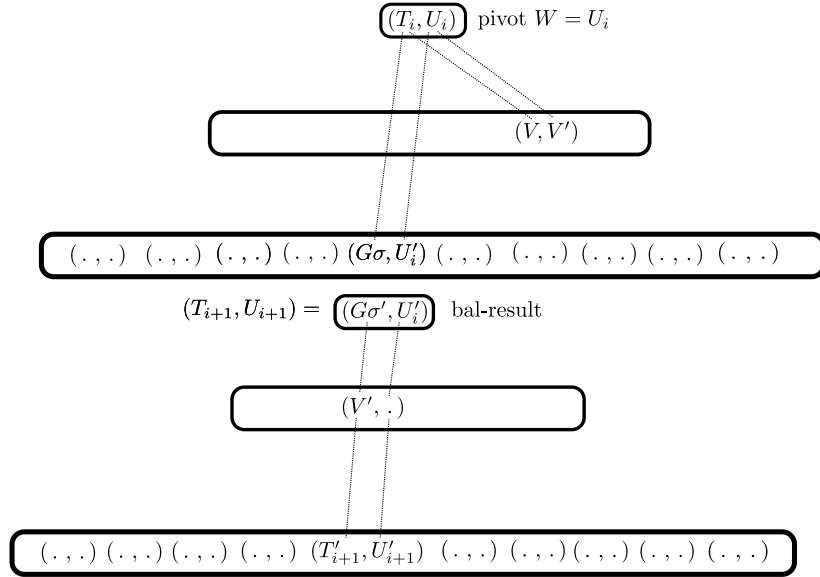


Fig. 4. Two consecutive rounds, with a left-balancing in the first one

We can easily verify that Prop. 4 holds also for the second game-version. The crucial point is that $\text{EqLv}(T_{i+1}, U_{i+1}) = \text{EqLv}(T'_i, U'_i)$ when Refuter uses the least-eqlevel strategy (this is based on Prop. 6(2) and Prop. 5).

Bal-results are close to pivots. When doing a left-balancing, replacing $(T'_i, U'_i) = (G\sigma, U'_i)$ with the *bal-result* $(T_{i+1}, U_{i+1}) = (G\sigma', U'_i)$, we might not have $U_i \rightsquigarrow_k (T_{i+1}, U_{i+1})$ for the *pivot* U_i , but we surely have $U_i \overset{L:d}{\rightsquigarrow}_k (T_{i+1}, U_{i+1})$, for $d = \text{HEIGHT}(G)$, where we generally extend the notation from (1) as follows:

$$\begin{aligned} W \overset{L:d}{\rightsquigarrow}_k (T, U) &\Leftrightarrow_{\text{df}} \text{there is a finite term } G \text{ and a substitution } \sigma \\ &\text{such that } T = G\sigma, \text{HEIGHT}(G) \leq d, W \rightsquigarrow_k U, \text{ and} \\ &W \rightsquigarrow_k V \text{ for all } V \in \text{RANGE}(\sigma); \end{aligned} \quad (2)$$

the symbol L signals that we allow a special head in the *left*-hand component (here with the height at most d). Symmetrically we define $W \overset{R:d}{\rightsquigarrow}_k (T, U)$, where R refers to the right-hand component.

Two remaining steps in the proof of Theorem 1. We first give an informal sketch, which is then formalized. We recall that a play of the Prover-Refuter game gives rise to a sequence $(T_0, U_0), (T_1, U_1), (T_2, U_2), \dots$ of pairs of terms that are the starting pairs for the rounds $1, 2, 3, \dots$, respectively.

In the first of the remaining proof steps (captured by Lemma 7) we show that in the case $T_0 \sim U_0$ Prover can force a certain potentially infinite (n, g) -subsequence of $(T_1, U_1), (T_2, U_2), \dots$, by a simple balancing strategy.

In the second step (Lemma 10) we bound the lengths of *eqlevel-decreasing* (n, g) -sequences. It turns out that Prover can compute a respective bound $\ell_{n,g} \in \mathbb{N}$ on condition that she guesses correctly the pairs of equivalent terms up to a certain (large) presentation size.

In the final game-version Prover wins if the length of a created (n, g) -sequence exceeds $\ell_{n,g}$, but Refuter's least-eqlevel strategy will be still winning if Prover does not guess correctly when computing $\ell_{n,g}$.

Eqlevel-decreasing sequences, and (n, g) -sequences. A sequence $(V_1, V'_1), (V_2, V'_2), (V_3, V'_3), \dots$ of pairs of (regular) terms is *eqlevel-decreasing* if $\omega > \text{EqLv}(V_1, V'_1) > \text{EqLv}(V_2, V'_2) > \dots$. In this case the sequence must be finite, and our requirement $V_1 \not\sim V'_1$ implies that its length is bounded by $1 + \text{EqLv}(V_1, V'_1)$.

Given a pair (n, g) where $n \in \mathbb{N}$ and $g : \mathbb{N}_+ \rightarrow \mathbb{N}_+$ is a nondecreasing function (where $\mathbb{N}_+ = \{1, 2, \dots\}$), a (finite or infinite) sequence of pairs of terms is an (n, g) -sequence if it can be presented as

$$(E_1\sigma, F_1\sigma), (E_2\sigma, F_2\sigma), (E_3\sigma, F_3\sigma), \dots$$

for a substitution σ with $|\text{SUPP}(\sigma)| \leq n$, where $\text{PRSIZE}(E_j, F_j) \leq g(j)$ for $j = 1, 2, \dots$ (We put $\text{PRSIZE}(E, F) = \text{PRSIZE}(E) + \text{PRSIZE}(F)$, say.) Thus the growth of the (regular) “head-terms” E_j, F_j is bounded by the function g , while at most n fixed “tail-subterms” (of unrestricted size) suffice for this presentation.

Prover can force an (n, g) -subsequence by a balancing strategy. We aim to prove Lemma 7; the proof is the most technical part of the paper, and we thus first explain the idea informally. Prover will use a simple balancing strategy, when starting with $T_0 \sim U_0$:

- In each round Prover chooses $k = M_1$ (in the point 1 of the game), where M_1 is a sufficiently large constant computed from the grammar \mathcal{G} ; she also uses only $\mathcal{B}_j \subseteq \sim$, thus keeping $T_i \sim U_i$ for all i .
- Prover balances (in the point 3 of the second game-version), e.g. by replacing $(T'_i, U'_i) = (G\sigma, U'_i)$ with $(T_{i+1}, U_{i+1}) = (G\sigma', U'_i)$ in the case of left-balancing, *only when* the respective *special head G has a bounded height*, bounded by some sufficiently large M'_0 ; the above M_1 was chosen sufficiently larger than M'_0 .
- Obeying the above “bounded-head” condition, Prover balances in any round in which she has an opportunity, but she has still another constraint: *Prover does not “switch” balancing sides in two consecutive rounds*, i.e., if she does a left-balancing in the $(i+1)$ -th round, then she cannot do a right-balancing in the $(i+2)$ -th round, and vice versa.

To sketch the idea why this strategy enables to present an infinite subsequence of $(T_1, U_1), (T_2, U_2), \dots$ as an (n, g) -sequence, we first explore the case where Prover does a left-balancing in the $(i+1)$ -th round. Hence we have

$$W = U_i \overset{L: M'_0}{\rightsquigarrow}_{M_1} (T_{i+1}, U_{i+1}) = (G\sigma', U_{i+1})$$

(using the notation in (2)), where W is the respective pivot and $(G\sigma', U_{i+1})$ the respective bal-result; this is also illustrated in Fig. 4.

We have two possibilities for the following $(i+2)$ -th round:

1. There is a left-balancing in the $(i+2)$ -th round.
The pivot of this balancing is $W' = U_{i+1}$, and we have $W \xrightarrow{u} W'$ where $|u| = M_1$; hence W' is “boundedly reachable” from W in this case.
2. Left-balancing (with a bounded head) is not possible in the $(i+2)$ -th round.
Here we have “no change”, i.e., $(T_{i+2}, U_{i+2}) = (T'_{i+1}, U'_{i+1})$, and we will derive that $W \rightsquigarrow_{2M_1} (T_{i+2}, U_{i+2})$ (using the notation in (1)). Now the pivot W' of the first next balancing in future will be again reachable from W ; maybe not boundedly reachable from W but boundedly reachable from a subterm of W .

The claims in the case 2 are based on the fact that the impossibility to do a left-balancing in the $(i+2)$ -th round entails that the respective path $T_{i+1} = G\sigma' \xrightarrow{u} T'_{i+1}$ is a shortest path from $G\sigma'$ to T'_{i+1} and is steadily “sinking” (or “popping”), exposing deeper and deeper subterms of $G\sigma'$; our choice of M'_0 and M_1 will guarantee that $T_{i+1} = G\sigma' \xrightarrow{u} T'_{i+1}$ can be then written $G\sigma' \xrightarrow{u'} x_\ell \sigma' \xrightarrow{u''} T'_{i+1}$, thus “erasing” G and exposing some $x_\ell \sigma'$ that is reachable from W within M_1 steps; this is depicted in Fig. 4, where $x_\ell \sigma' = V'$.

A simple analysis now shows that if there is no repeat in the sequence $(T_0, U_0), (T_1, U_1), (T_2, U_2), \dots$, then there must be infinitely many balancing rounds, with the respective pivots denoted W_1, W_2, W_3, \dots , while each concrete pivot can repeat only boundedly many times. In the special “pivot path” $W_1 \xrightarrow{w_1} W_2 \xrightarrow{w_2} W_3 \xrightarrow{w_3} \dots$ which we touched on (recall that W_{j+1} is boundedly reachable from a subterm of W_j) we then must have a deepest subterm V_0 of W_1 ,

visited in some segment $W_{j_0} \xrightarrow{w_{j_0}} W_{j_0+1}$, written as $W_{j_0} \xrightarrow{w'_{j_0}} V_0 \xrightarrow{w''_{j_0}} W_{j_0+1}$, such that the path $V_0 \xrightarrow{w'_{j_0}} W_{j_0+1} \xrightarrow{w_{j_0+1}} W_{j_0+2} \xrightarrow{w_{j_0+2}} \dots$ does not visit any subterm of V_0 . Then the sequence of bal-results related to $W_{j_0+1}, W_{j_0+2}, W_{j_0+3}, \dots$ can be presented as an (n, g) -sequence, where n, g are determined by \mathcal{G} .

The proof of the next lemma just makes clear all relevant technical details.

Lemma 7. *There are n_0, g_0 ($n_0 \in \mathbb{N}$, $g_0 : \mathbb{N}_+ \rightarrow \mathbb{N}_+$) determined by (in fact, computable from) grammar \mathcal{G} such that Prover can force for any initial $T_0 \sim U_0$ that she either wins or the sequence $(T_1, U_1), (T_2, U_2), \dots$ has an infinite (n_0, g_0) -subsequence.*

Proof. We first introduce some technical notions related to a given grammar $\mathcal{G} = (\mathcal{N}, \Sigma, \mathcal{R})$; in our notation we assume that $\text{arity}(A) = m$ for all $A \in \mathcal{N}$.

If $A(x_1, \dots, x_m) \xrightarrow{w} x_i$ in $\mathcal{L}_{\mathcal{G}}^R$, then we call $w \in \mathcal{R}^*$ an (A, i) -sink word. We assume that for each pair $A \in \mathcal{N}$, $i \in [1, m]$ there is a fixed shortest (A, i) -sink word $w_{[A, i]}$, and we put

$$M_0 = 1 + \max \{ |w_{[A, i]}|; A \in \mathcal{N}, i \in [1, m] \}.$$

The words $w_{[A, i]}$ can be found and M_0 can be computed by a standard dynamic programming approach.

We note that $|w_{[A, i]}| = 1$ if there is a rule $A(x_1, \dots, x_m) \xrightarrow{a} x_i$; otherwise $|w_{[A, i]}| = 1 + |u|$ where u is a shortest word such that $E \xrightarrow{u} x_i$ for a rule $A(x_1, \dots, x_m) \xrightarrow{a} E$; moreover, the path $E \xrightarrow{u} x_i$ “sinks” along a branch in E till a leaf x_i , and can be composed from the relevant shorter words $w_{[B, j]}$. Though M_0 can be exponential (as demonstrated by the rules $A_k(x_1) \xrightarrow{a} A_{k-1}(A_{k-1}(x_1)), A_{k-1}(x_1) \xrightarrow{a} A_{k-2}(A_{k-2}(x_1)), \dots, A_2(x_1) \xrightarrow{a} A_1(A_1(x_1)), A_1(x_1) \xrightarrow{a} x_1$), it can be computed in polynomial time.

If we find that there are no (A, i) -sink words for some A, i , then the i -th root-successor of any $A(G_1, \dots, G_m)$ plays no role (i.e., its replacing does not change the equivalence class); we can then simply omit the i -th root-successors when the root is A . We can thus decrease $\text{arity}(A)$, and modify the grammar rules accordingly so that the LTSs $\mathcal{L}_{\mathcal{G}}^R$ and $\mathcal{L}_{\mathcal{G}}^A$ do not change, up to isomorphism. We can thus indeed safely assume that there are $w_{[A, i]}$ for all $A \in \mathcal{N}$, $i \in [1, \text{arity}(A)]$.

A path $V \xrightarrow{u}$ in $\mathcal{L}_{\mathcal{G}}^R$ is *root-performable*, if $A(x_1, \dots, x_m) \xrightarrow{u}$ where $A = \text{root}(V)$ (in which case u is enabled by any term with the root A). A path $V \xrightarrow{w}$ in $\mathcal{L}_{\mathcal{G}}^R$ is a *non-sink segment*, a *non-sink* for short, if $|w| = M_0$ and $V \xrightarrow{w}$ is root-performable. (For each root-successor V' in V we thus have $V \xrightarrow{v} V'$ for some v shorter than w .)

A path $T \xrightarrow{u} T'$ in $\mathcal{L}_{\mathcal{G}}^R$ is *sinking* if it contains no non-sink, i.e., for any partition $u = u_1 u_2 u_3$ with $|u_2| = M_0$ we have $u_2 = u'_2 u''_2$ ($u'_2 \neq \varepsilon$) where $T \xrightarrow{u_1} V' \xrightarrow{u'_2} V'' \xrightarrow{u''_2 u_3} T'$ and V'' is a root-successor in V' . Hence if $T \xrightarrow{u} T'$

is sinking, then it can be written $T \xrightarrow{u_1} V \xrightarrow{u_2} T'$ where $|u_2| < M_0$ and V is a subterm of T in depth at least $|u| \div M_0$. (By \div we denote integer division.)

Finally we consider a *shortest path* $T \xrightarrow{u} T'$ from T to T' that is not sinking. It can be written $T \xrightarrow{u_1} V \xrightarrow{u_2} V' \xrightarrow{u_3} T'$ where $V \xrightarrow{u_2} V'$ is the last non-sink. We can easily check that then $T' = G\sigma$ where $\text{HEIGHT}(G) \leq M'_0$, for some M'_0 determined by \mathcal{G} , and $\text{RANGE}(\sigma)$ consists of the root-successors in V .

To verify the claim, we first note that we cannot have $T \xrightarrow{u_1} V \xrightarrow{u_2} V' \xrightarrow{u_3} V'' \xrightarrow{u_{32}} T'$ where V'' is a root-successor in V , since there would be a shorter path $T \xrightarrow{u_1} V \xrightarrow{w} V'' \xrightarrow{u_{32}} T'$ from T to T' (for w being the relevant sink-word $w_{[A,i]}$, satisfying $|w| < M_0$). Hence $V \xrightarrow{u_2 u_3} T'$ is root-performable, and we have $V = (A(x_1, \dots, x_m))\sigma \xrightarrow{u_2 u_3} T' = G\sigma$ where $A(x_1, \dots, x_m) \xrightarrow{u_2 u_3} G$ and $\text{RANGE}(\sigma)$ consists of the root-successors in V . Since we took the last non-sink in $T \xrightarrow{u} T'$, the path $V \xrightarrow{u_2 u_3} T'$, and thus also $A(x_1, \dots, x_m) \xrightarrow{u_2 u_3} G$, is sinking after its first step. Therefore G is reachable within less than M_0 steps from a subterm of the rhs E of a rule $A(x_1, \dots, x_m) \xrightarrow{a} E$ in the set \mathcal{R} of rules in the grammar \mathcal{G} . We can thus (generously) put $M'_0 = (1 + M_0) \cdot \text{MAXRH}$ where $\text{MAXRH} = \max \{ \text{HEIGHT}(F) \mid \text{there is a rule } B(x_1, \dots, x_m) \xrightarrow{a} F \text{ in } \mathcal{R} \}$.

We now take M_1 such that $(M_1 \div M_0) > M'_0$, and show Prover's strategy in the $(i+1)$ -th round, when starting with $T_i \sim U_i$:

- i/ Prover chooses $k = M_1$ and $\{(T_i, U_i)\} = \mathcal{B}_0 \triangleleft \mathcal{B}_1 \triangleleft \mathcal{B}_2 \triangleleft \dots \triangleleft \mathcal{B}_{M_1}$ where $\mathcal{B}_j \subseteq \sim$ (for all $j \in [1, M_1]$). Refuter chooses $(T'_i, U'_i) \in \mathcal{B}_{M_1}$ and we can fix some paths $T_i \xrightarrow{u_1} T'_i$, $U_i \xrightarrow{u_2} U'_i$ in $\mathcal{L}_{\mathcal{G}}^R$, where $|u_1| = |u_2| = M_1$, $\text{ACT}(u_1) = \text{ACT}(u_2)$ (recall again Fig. 3).
- ii/ If $T_i \xrightarrow{u_1} T'_i$ is not a shortest path from T_i to T'_i or contains a non-sink, and Prover did not do a right-balancing in the (previous) i -th round, then she makes a left-balancing, replacing $(T'_i, U'_i) = (G\sigma, U'_i)$ with $(T_{i+1}, U_{i+1}) = (G\sigma', U'_i)$, for some head G with the smallest possible height. (We know that $\text{HEIGHT}(G) \leq M'_0$.)
- iii/ If ii/ did not apply, and $U_i \xrightarrow{u_2} U'_i$ is not shortest or contains a non-sink, and Prover did not do a left-balancing in the i -th round, then she makes a right-balancing, symmetrically to ii/.
- iv/ If none of ii/, iii/ applied, Prover puts $(T_{i+1}, U_{i+1}) = (T'_i, U'_i)$.

Before analysing the outcome of the strategy, we recall that each bal-result (T_{i+1}, U_{i+1}) has its pivot W , where $W \xrightarrow{\text{L: } M'_0} M_1 (T_{i+1}, U_{i+1})$ or $W \xrightarrow{\text{R: } M'_0} M_1 (T_{i+1}, U_{i+1})$ (recall the definition in (2)), and we explore the case when Prover does a left-balancing in the $(i+1)$ -th round, with the pivot $W = U_i$, but she cannot do a left-balancing (and thus any balancing) in the $(i+2)$ -th round, as depicted in Fig. 4. (We omit the case with a right-balancing, since it is symmetric.)

In the mentioned case we have $W \xrightarrow{\text{L: } M'_0} M_1 (T_{i+1}, U_{i+1}) = (G\sigma', U_{i+1})$, where $\text{HEIGHT}(G) \leq M'_0$ and each $V' \in \text{RANGE}(\sigma')$ is reachable from W within M_1 steps. Now the respective path $T_{i+1} = G\sigma' \xrightarrow{u} T'_{i+1}$ (created in the $(i+2)$ -th

round) is sinking (and shortest). But then $W \rightsquigarrow_{2M_1} (T_{i+2}, U_{i+2})$ as can be easily verified.

Indeed, we have chosen M_1 large enough ($(M_1 \div M_0) > M'_0 \geq \text{HEIGHT}(G)$) so that the sinking path $G\sigma' \xrightarrow{u} T'_{i+1}$ can be written $G\sigma' \xrightarrow{u'} x_\ell\sigma' \xrightarrow{u''} T'_{i+1}$ (where $G \xrightarrow{u'} x_\ell$); informally, the path sinks along a branch of G until a leaf x_ℓ of G (where $x_\ell\sigma'$ hangs). Since $x_\ell\sigma'$ is reachable from W within M_1 steps, we have $W \xrightarrow{v} x_\ell\sigma' \xrightarrow{u''} T'_{i+1}$ where $|vu''| \leq 2M_1$. On the other hand, our definitions yield that $W = U_i \xrightarrow{v_1} U_{i+1} \xrightarrow{v_2} U_{i+2}$ for some words v_1, v_2 where each has the length M_1 .

We now explore an infinite play from $T_0 \sim U_0$ where Prover uses the above strategy. We first note that there are *infinitely many balancings*; otherwise from some round on we would have constant sinking on both sides, which necessarily leads to a repeat since our terms are regular.

Suppose we have $T_i \xrightarrow{u_i} T_{i+1} \xrightarrow{u_{i+1}} T_{i+2} \xrightarrow{u_{i+2}} \dots$ where all paths $T_{i+j} \xrightarrow{u_{i+j}} T_{i+j+1}$ (each of length M_1) are sinking. Then T_{i+1} is reachable from a subterm of T_i in less than M_0 steps, and we can thus write $T_{i+1} = G\sigma$ where $\text{HEIGHT}(G) \leq M'_0$ and all $V \in \text{RANGE}(\sigma)$ are subterms of T_i . Then the path $T_{i+1} = G\sigma \xrightarrow{u_{i+1}} T_{i+2}$ first sinks along a branch of G until exposing a subterm of T_i ; hence T_{i+2} is also reachable from a subterm of T_i in less than M_0 steps. Inductively we thus derive that each T_{i+j} is reachable from a subterm of T_i in less than M_0 steps, hence all $T_i, T_{i+1}, T_{i+2}, \dots$ range over a finite set. (Similarly $U_i, U_{i+1}, U_{i+2}, \dots$ would range over a finite set when there were only finitely many balancings.)

We denote the pivots of our infinitely many balancings by W_1, W_2, W_3, \dots , and we easily verify that for each j we have a path $W_j \xrightarrow{w_j} W_{j+1}$ (in \mathcal{L}_G^R) of the form

$$W_j \xrightarrow{v_1} V' \xrightarrow{v_2} V'' \xrightarrow{v_3} W_{j+1} \quad (3)$$

where $|v_1|, |v_3|$ are bounded (surely by $2M_1$) and V'' is a subterm of V' ; though v_2 can be sometimes long (and sometimes empty), we can assume the path $V' \xrightarrow{v_2} V''$ to be sinking.

In the case of balancings in two consecutive rounds (which are then both left-balancings, or both right-balancings), with pivots W_j and W_{j+1} , we have $W_j \xrightarrow{u} W_{j+1}$ where $|u| = M_1$. Suppose now two consecutive balancings, with pivots $W_j \in \{T_{i_1}, U_{i_1}\}$ and $W_{j+1} \in \{T_{i_2}, U_{i_2}\}$ that did not happen in two consecutive rounds, hence $i_2 \geq i_1 + 2$. By our above analysis we have $W_j \rightsquigarrow_{2M_1} (T_{i_1+2}, U_{i_1+2})$, and the strategy implies that we have either $T_{i_1+2} \xrightarrow{u_3} T_{i_1+3} \xrightarrow{u_4} T_{i_1+4} \xrightarrow{u_5} \dots \xrightarrow{u_{i_2}} T_{i_2} = W_{j+1}$ or $U_{i_1+2} \xrightarrow{u_3} U_{i_1+3} \xrightarrow{u_4} U_{i_1+4} \xrightarrow{u_5} \dots \xrightarrow{u_{i_2}} U_{i_2} = W_{j+1}$ where each (sub)path $T_{i_1+\ell-1} \xrightarrow{u_\ell} T_{i_1+\ell}$, or $U_{i_1+\ell-1} \xrightarrow{u_\ell} U_{i_1+\ell}$, has length M_1 and is sinking. Similarly as previously, we derive that W_{j+1} is reachable in less than M_0 steps from a subterm of either T_{i_1+2} or U_{i_1+2} .

Suppose now that the “pivot path”

$$W_1 \xrightarrow{w_1} W_2 \xrightarrow{w_2} W_3 \xrightarrow{w_3} \dots$$

visits subterms of W_1 infinitely often. Then the pivots W_j are infinitely often boundedly reachable from a subterm of W_1 , as follows from the form (3) of paths $W_j \xrightarrow{w_j} W_{j+1}$. In this case one pivot reappears infinitely often; but there are boundedly many bal-results for one pivot, and we would thus have a repeat.

Recall that the bal-result (T, U) related to pivot W satisfies $W \xrightarrow{L: M'_0}_{M_1} (T, U)$ or $W \xrightarrow{R: M'_0}_{M_1} (T, U)$ (as defined in (2)); hence we have boundedly many possible (T, U) for one W .

Some segment $W_{j_0} \xrightarrow{w_{j_0}} W_{j_0+1}$ thus visits a subterm of W_1 , denoted by V_0 , for the last time. Hence $W_{j_0} \xrightarrow{w'_{j_0}} V_0 \xrightarrow{w''_{j_0}} W_{j_0+1}$, and the infinite path $V_0 \xrightarrow{w''_{j_0}} \xrightarrow{w_{j_0+1}} \xrightarrow{w_{j_0+2}} \dots$ is root-performable; for $A = \text{ROOT}(V_0)$ we have

$$A(x_1, \dots, x_m) \xrightarrow{w''_{j_0}} G_1 \xrightarrow{w_{j_0+1}} G_2 \xrightarrow{w_{j_0+2}} G_3 \xrightarrow{w_{j_0+3}} \dots \quad (4)$$

Hence $V_0 = (A(x_1, \dots, x_m))\sigma'$ and $W_{j_0+\ell} = G_\ell\sigma'$ ($\ell = 1, 2, \dots$) for σ' whose range consists of the root-successors in V_0 . We also note that $\text{HEIGHT}(G_\ell)$ can only boundedly grow (with growing ℓ).

By the form of the paths (3), we know that $G_{\ell+1}$ is boundedly reachable from a subterm of G_ℓ ; to be more precise, $G_{\ell+1}$ is reachable within M_0 steps from a subterm of a term that is reachable within $2M_1$ steps from G_ℓ . Hence we surely have $\text{HEIGHT}(G_{\ell+1}) \leq \text{HEIGHT}(G_\ell) + (2M_1 + M_0) \cdot \text{MAXRH}$ where $\text{MAXRH} = \max \{ \text{HEIGHT}(F) \mid \text{there is a rule } B(x_1, \dots, x_m) \xrightarrow{a} F \text{ in } \mathcal{R} \}$. Since $\text{HEIGHT}(G_1) \leq 1 + (2M_1 + M_0) \cdot \text{MAXRH}$, we have $\text{HEIGHT}(G_\ell) \leq 1 + \ell \cdot (2M_1 + M_0) \cdot \text{MAXRH}$.

We are interested in the bal-results related to $W_{j_0+1}, W_{j_0+2}, W_{j_0+3}, \dots$, i.e., to $G_1\sigma', G_2\sigma', G_3\sigma', \dots$. Since the bal-result (T, U) related to $G_\ell\sigma'$ satisfies $G_\ell\sigma' \xrightarrow{L: M'_0}_{M_1} (T, U)$ or $G_\ell\sigma' \xrightarrow{R: M'_0}_{M_1} (T, U)$, it is built from some finite bounded “head-terms”, and some “tail-terms” that are subterms of $G_\ell\sigma'$ in depth at most M_1 .

A path $W \xrightarrow{v} V$ obviously cannot “expose”, i.e. “sink to”, a subterm of W that is deeper than $|v|$.

It is useful to rather write

$$V_0 = (A(x_1, \dots, x_m))\sigma' = F\sigma \quad (5)$$

for a finite term F in which each branch has length M_1 if it is not a complete branch of V_0 , and where $\text{RANGE}(\sigma)$ consists of the subterms of V_0 with depth M_1 .

To get F and σ , for each particular occurrence of a subterm U of V_0 that has depth M_1 in V_0 we do the following: we replace this occurrence of U with a fresh variable x_i and we put $\sigma(x_i) = U$. The resulting term F is a finite term with $\text{HEIGHT}(F) \leq M_1$, and $\text{SUPP}(\sigma)$ consists of at most m^{M_1} variables, where m is the maximum arity of nonterminals of the grammar \mathcal{G} . Putting

$$n_0 = m^{M_1},$$

we get $|\text{SUPP}(\sigma)| \leq n_0$.

Recalling (4) and (5), we have

$$F \xrightarrow{w_{j_0}''} H_1 \xrightarrow{w_{j_0}+1} H_2 \xrightarrow{w_{j_0}+2} H_3 \xrightarrow{w_{j_0}+3} \dots$$

where $W_{j_0+\ell} = G_\ell \sigma' = H_\ell \sigma$, $\text{HEIGHT}(H_\ell) \leq \text{HEIGHT}(G_\ell) + M_1$, and each occurrence of $x_i \in \text{SUPP}(\sigma)$ in H_ℓ has depth at least M_1 (for $\ell = 1, 2, \dots$). The bal-result (T, U) related to $W_{j_0+\ell} = H_\ell \sigma$ (satisfying $H_\ell \sigma \overset{\text{L}: M'_0}{\rightsquigarrow}_{M_1} (T, U)$ or $H_\ell \sigma \overset{\text{R}: M'_0}{\rightsquigarrow}_{M_1} (T, U)$) can be thus written

$$(T, U) = (E_\ell \sigma, F_\ell \sigma)$$

for finite terms E_ℓ, F_ℓ whose height, and thus also size, can only boundedly grow with growing ℓ (since $\text{HEIGHT}(H_\ell)$ can only boundedly grow with growing ℓ).

If $H_\ell \sigma \xrightarrow{v} V$ (in $\mathcal{L}_\mathcal{G}^\mathbb{R}$) where $|v| \leq M_1$, then $H_\ell \xrightarrow{v} H'$ where $V = H' \sigma$, since the subterm-occurrences with depth at least M_1 in $H_\ell \sigma$ need at least M_1 steps for being exposed. Moreover, $\text{HEIGHT}(H') \leq \text{HEIGHT}(H_\ell) + M_1 \cdot \text{MAXRH}$ (where MAXRH bounds the height-increase in one step).

Recall that $H_\ell \sigma \overset{\text{L}: M'_0}{\rightsquigarrow}_{M_1} (T, U)$ entails $H_\ell \sigma \rightsquigarrow_{M_1} U$, and $T = G \sigma''$ where $\text{HEIGHT}(G) \leq M'_0$ and $H_\ell \sigma \rightsquigarrow_{M_1} V$ for each $V \in \text{RANGE}(\sigma'')$. Hence each term from the set $\{U\} \cup \{V \mid V \in \text{RANGE}(\sigma'')\}$ can be written in the form $E' \sigma$ for some E' with $\text{HEIGHT}(E') \leq \text{HEIGHT}(H_\ell) + M_1 \cdot \text{MAXRH}$. Therefore we can write $U = F_\ell \sigma$ and $T = G \sigma''' \sigma$ where the height of F_ℓ and of each $E' \in \text{RANGE}(\sigma''')$ is bounded by $\text{HEIGHT}(H_\ell) + M_1 \cdot \text{MAXRH}$. Finally we put $E_\ell = G \sigma'''$. Hence $(T, U) = (E_\ell \sigma, F_\ell \sigma)$, and we surely have $\text{PRSIZE}(E_\ell, F_\ell) \leq 2 \cdot (m^H)^2$ where m is the maximal arity of nonterminals and $H = M'_0 + \text{HEIGHT}(H_\ell) + M_1 \cdot \text{MAXRH}$. Since $\text{HEIGHT}(H_\ell) \leq \text{HEIGHT}(G_\ell) + M_1$ and $\text{HEIGHT}(G_\ell) \leq 1 + \ell \cdot (2M_1 + M_0) \cdot \text{MAXRH}$, we get

a function g_0 , determined by the grammar \mathcal{G} ,

for which $\text{PRSIZE}(E_\ell, F_\ell) \leq g_0(\ell)$, for $\ell = 1, 2, 3, \dots$

Hence the bal-results related to the pivots $W_{j_0+1}, W_{j_0+2}, W_{j_0+3}, \dots$, i.e., to $H_1 \sigma, H_2 \sigma, H_3 \sigma, \dots$, can be presented as an (n_0, g_0) -sequence

$$(E_1 \sigma, F_1 \sigma), (E_2 \sigma, F_2 \sigma), (E_3 \sigma, F_3 \sigma), \dots,$$

where n_0, g_0 that are determined by the grammar \mathcal{G} . □

The lengths of eqlevel-decreasing (n, g) -sequences are bounded. Before proving Lemma 10 we show some useful facts and convenient notions, assuming a grammar $\mathcal{G} = (\mathcal{N}, \Sigma, \mathcal{R})$. We first recall that $\text{EqLv}(E, F) \leq \text{EqLv}(E\sigma, F\sigma)$, and note:

Proposition 8. *If $\text{EqLv}(E, F) = k < e = \text{EqLv}(E\sigma, F\sigma)$ (where $e \in \mathbb{N} \cup \{\omega\}$) then there are $x_i \in \text{SUPP}(\sigma)$, $H \neq x_i$, and $w \in \Sigma^*$, where $|w| \leq k$, such that $E \xrightarrow{w} x_i$, $F \xrightarrow{w} H$ or $E \xrightarrow{w} H$, $F \xrightarrow{w} x_i$, and $x_i\sigma \sim_{e-k} H\sigma$.*

Proof. We take $\{(E\sigma, F\sigma)\} = \mathcal{B}_0 \triangleleft \mathcal{B}_1 \triangleleft \dots \triangleleft \mathcal{B}_{k+1}$, so that $\text{MINEQL}(\mathcal{B}_j) = e - j$ for all $j \in [0, k+1]$. (Recall Prop. 3.) When trying to mimic this sequence by replacing σ with the empty-support substitution and aiming to create $\{(E, F)\} = \mathcal{B}'_0 \triangleleft \mathcal{B}'_1 \triangleleft \dots \triangleleft \mathcal{B}'_{k+1}$, we must get (x_i, H) or (H, x_i) with $H \neq x_i$ in some \mathcal{B}'_j for $j \leq k$ (instead of the original pair $(x_i\sigma, H\sigma)$ or $(H\sigma, x_i\sigma)$), since otherwise we would prove $E \sim_{k+1} F$. Since $\text{EqLv}(x_i\sigma, H\sigma) \geq \text{MINEQL}(\mathcal{B}_j)$ for the respective $j \leq k$, we surely have $x_i\sigma \sim_{e-k} H\sigma$. \square

By $\{(x_i, H)\}$ we denote the substitution that (only) replaces x_i with H (i.e., $x_i\{(x_i, H)\} = H$ and $x_j\{(x_i, H)\} = x_j$ for $j \neq i$). Hence $\{(x_i, H)\}\sigma$ is the substitution σ' satisfying $x_i\sigma' = H\sigma$ and $x_j\sigma' = x_j\sigma$ for all $j \neq i$. We note that the (“limit” regular) term

$$H' = H\{(x_i, H)\}\{(x_i, H)\} \dots \quad (6)$$

is well defined and satisfies $H' = H\{(x_i, H')\}$: a graph presentation of H' arises from a graph presentation of H by redirecting each arc leading to x_i (if there is any) towards the root. (We have $H' = H$ if x_i does not occur in H , or if $H = x_i$.) Hence also $\text{PrSize}(H') \leq \text{PrSize}(H)$. E.g., for the terms in Fig. 1 we have $E_2 = E_1\{(x_2, E_1)\}\{(x_2, E_1)\}\{(x_2, E_1)\} \dots = E_1\{(x_2, E_2)\}$.

By $\sigma_{[-x_i]}$ we denote the substitution arising from σ by removing x_i from the support (if it is there), i.e.,

$$x_i\sigma_{[-x_i]} = x_i \text{ and } x_j\sigma_{[-x_i]} = x_j\sigma \text{ for } j \neq i.$$

If $H \neq x_i$, then x_i does not occur in H' defined by (6); we then have $H'\sigma = H'\sigma_{[-x_i]}$, and this enables an inductive argument in the proof of Lemma 10, based on stepwise decreasing the substitution support (i.e., the number n in eqlevel-decreasing (n, g) -sequences).

Recalling that $\sigma \sim_k \sigma'$ iff $x_j\sigma \sim_k x_j\sigma'$ for all $x_j \in \text{VAR}$, and referring to H' in (6), we also note the following fact (which follows from the congruence properties, by a repeated use of Prop. 6(2)):

Proposition 9. *If $x_i\sigma \sim_k H\sigma$ and $H \neq x_i$, then $\sigma \sim_k \{(x_i, H')\}\sigma_{[-x_i]}$.*

Proof. Assume $H \neq x_i$; hence x_i does not occur in H' , and we also recall that $H' = H\{(x_i, H')\}$, and $H'\sigma = H'\sigma_{[-x_i]}$.

For $j \neq i$ we obviously have $x_j\sigma = x_j\{(x_i, H')\}\sigma = x_j\{(x_i, H')\}\sigma_{[-x_i]}$. Hence we will be done if we show that $\text{EqLv}(x_i\sigma, H\sigma) = \text{EqLv}(x_i\sigma, x_i\{(x_i, H')\}\sigma_{[-x_i]})$, i.e., if we show that

$$\text{EqLv}(x_i\sigma, H\sigma) = \text{EqLv}(x_i\sigma, H'\sigma). \quad (7)$$

Let $\text{EqLv}(H\sigma, H'\sigma) = e$; this can be also written $\text{EqLv}(H\sigma, H\{(x_i, H')\}\sigma) = e$.

If $e = \omega$, then (7) is clear. If $e < \omega$, then

$$e = \text{EqLv}(H\sigma, H\{(x_i, H')\}\sigma) > \text{EqLv}(\sigma, \{(x_i, H')\}\sigma) = \text{EqLv}(x_i\sigma, H'\sigma)$$

(where the inequality “ $>$ ” follows from Prop. 6(2)). Thus $\text{EqLv}(H\sigma, H'\sigma) > \text{EqLv}(x_i\sigma, H'\sigma)$, and (7) follows by Prop. 5. \square

Remark. We discussed a possible decomposition approach after noting the congruence properties (Prop. 6). Now Propositions 8 and 9 also suggest a certain decomposition approach, as we now sketch.

Suppose we have $T \sim U$. We can present (T, U) as $(E\sigma, F\sigma)$ in many ways.

E.g., if $A = \text{root}(T)$ and $B = \text{root}(U)$ then we can put $E = A(x_1, \dots, x_m)$, $F = B(x_{m+1}, \dots, x_{2m})$ (assuming $\text{arity}(A) = \text{arity}(B) = m$), and for $i \in [1, m]$ we define $x_i\sigma$ to be the i -th root-successor in T and $x_{i+m}\sigma$ to be the i -th root-successor in U .

For $(T, U) = (E\sigma, F\sigma)$ where $T \sim U$ there are two possibilities:

1. either $E \sim F$, in which case $E\sigma' \sim F\sigma'$ for any σ' ,
2. or $\text{EqLv}(E, F) = k < \omega$.

In the case 2 we must have $x_i \in \text{SUPP}(\sigma)$ and $H \neq x_i$, where $E \rightsquigarrow_k H$ or $F \rightsquigarrow_k H$, such that

$$\sigma \sim \{(x_i, H')\}\sigma_{[-x_i]}$$

(by Prop. 8 and 9), and thus $E'\sigma_{[-x_i]} \sim F'\sigma_{[-x_i]}$ where

$$E' = E\{(x_i, H')\} \text{ and } F' = F\{(x_i, H')\}.$$

We can even bound the size of H , and thus of H' , as follows: $\text{PRSize}(H') \leq \text{PRSize}(E, F) + k \cdot \text{STEPINC}$, where STEPINC is defined in (8).

We also note that for any E, F, σ, x_i, H where $H \neq x_i$ (and maybe $E\sigma \not\sim F\sigma$), we have

$$\text{EqLv}(E\sigma, F\sigma) \geq \text{MINEqLv}(\{(x_i\sigma, H'\sigma), (E'\sigma_{[-x_i]}, F'\sigma_{[-x_i]})\}),$$

which can lead to a decomposition if the pairs $(x_i\sigma, H'\sigma)$, $(E'\sigma_{[-x_i]}, F'\sigma_{[-x_i]})$ are somehow “smaller” than $(E\sigma, F\sigma)$.

Moreover, in the case $E'\sigma_{[-x_i]} \sim F'\sigma_{[-x_i]}$ we can continue in the same way as above:

1. either $E' \sim F'$, in which case $E'\sigma' \sim F'\sigma'$ for any σ' ,
2. or $\text{EqLv}(E', F') = k' < \omega$.

In the latter case we get some $x_j \in \text{SUPP}(\sigma_{[-x_i]})$ and some G such that $E' \rightsquigarrow_{k'} G$ or $F' \rightsquigarrow_{k'} G$, and $E''((\sigma_{[-x_i]})_{[-x_j]}) \sim F''((\sigma_{[-x_i]})_{[-x_j]})$ where $E'' = E'\{(x_j, G')\}$ and $F'' = F'\{(x_j, G')\}$, for $G' = G\{(x_j, G)\}\{(x_j, G)\} \dots$.

Continuing this reasoning, we must come to the case 1 after at most n iterations where $n = |\text{SUPP}(\sigma)|$, maybe with the empty-support substitution in the end.

A problem is to define an adequate size of the pairs, to transform the above observations into a sound algorithm based on the respective decompositions. In the algorithm based on our Prover-Refuter game we circumvent this problem; we use the above observations for a (conditional, nondeterministic) computation of a bound on eqlevel-decreasing (n, g) -sequences.

We still add a few technical notions, useful for proving Lemma 10. For our assumed grammar $\mathcal{G} = (\mathcal{N}, \Sigma, \mathcal{R})$ we put

$$\text{STEPINC} = \max \{ \text{PRSIZE}(E) \mid E \text{ is the rhs of a rule in } \mathcal{G} \}. \quad (8)$$

We note that $F \xrightarrow{w} G$ implies $\text{PRSIZE}(G) \leq \text{PRSIZE}(F) + |w| \cdot \text{STEPINC}$.

For any set $\mathcal{B} \subseteq \text{TERMS}_{\mathcal{N}} \times \text{TERMS}_{\mathcal{N}}$ we put

$$\text{MAXEQL}(\mathcal{B}) = \max \{ \text{EQLV}(E, F) \mid (E, F) \in \mathcal{B} \},$$

stipulating $\max \emptyset = 0$. ($\text{MINEQL}(\mathcal{B})$ has been already defined.)

For any $b \in \mathbb{N}$, we put

$$\begin{aligned} \text{SIZE}_{\leq b} &= \{ (E, F) \mid \text{PRSIZE}(E, F) \leq b \}, \text{ and} \\ \text{MEL}_b &= \text{MAXEQL}(\text{SIZE}_{\leq b} \cap \not\sim). \end{aligned}$$

We note that MEL_b is always a *finite* number.

For any $n \in \mathbb{N}$ and $g : \mathbb{N}_+ \rightarrow \mathbb{N}_+$ we define $\ell_{n,g} \in \mathbb{N}$ by the following recursive definition:

$$\begin{aligned} \ell_{0,g} &= 1 + \text{MEL}_{g(1)}, \text{ and} \\ \ell_{n+1,g} &= 1 + \text{MEL}_{g(1)} + \ell_{n,g'} \text{ where} \end{aligned}$$

$$g'(j) = g(1 + \text{MEL}_{g(1)} + j) + 2 \cdot (g(1) + \text{MEL}_{g(1)} \cdot \text{STEPINC}) \text{ for all } j \in \mathbb{N}_+. \quad (9)$$

Lemma 10. *Any eqlevel-decreasing (n, g) -sequence has length at most $\ell_{n,g}$.*

Proof. By induction on n . Assume an eqlevel-decreasing (n, g) -sequence

$$(E_1\sigma, F_1\sigma), (E_2\sigma, F_2\sigma), \dots, (E_\ell\sigma, F_\ell\sigma),$$

which also entails $E_1\sigma \not\sim F_1\sigma$ by our definition. Since $\text{EQLV}(E_1, F_1) \leq \text{EQLV}(E_1\sigma, F_1\sigma)$, we have $E_1 \not\sim F_1$; moreover, $\text{EQLV}(E_1, F_1) \leq \text{MEL}_{g(1)}$ since $\text{PRSIZE}(E_1, F_1) \leq g(1)$.

If $n = 0$, then $(E_1, F_1) = (E_1\sigma, F_1\sigma)$, and thus

$$\ell \leq 1 + \text{EQLV}(E_1, F_1) \leq 1 + \text{MEL}_{g(1)} = \ell_{0,g};$$

we also have $\ell \leq \ell_{0,g}$ if $\text{EQLV}(E_1, F_1) = \text{EQLV}(E_1\sigma, F_1\sigma)$.

If $\text{EQLV}(E_1, F_1) = k < e = \text{EQLV}(E_1\sigma, F_1\sigma)$, then

$$\sigma \sim_{e-k} \{ (x_i, H') \} \sigma_{[-x_i]}$$

for some $x_i \in \text{SUPP}(\sigma)$ and some H' with $\text{PRSIZE}(H') \leq g(1) + k \cdot \text{STEPINC} \leq g(1) + \text{MEL}_{g(1)} \cdot \text{STEPINC}$; this can be easily derived from Prop. 8 and 9.

We now put (*shift*) $s = 1 + \text{MEL}_{g(1)}$; hence $s > \text{EQLV}(E_1, F_1) = k$, and thus

$$e - k > \text{EQLV}(E_{s+1}\sigma, F_{s+1}\sigma) > \dots > \text{EQLV}(E_{s+(\ell-s)}\sigma, F_{s+(\ell-s)}\sigma).$$

For $j = 1, 2, \dots, \ell - s$ we define

$$(E'_j, F'_j) = (E_{s+j}\{ (x_i, H') \}, F_{s+j}\{ (x_i, H') \}).$$

Since $\text{EqLv}(E'_j\sigma, E_{s+j}\sigma) = \text{EqLv}(E_{s+j}\sigma, E_{s+j}\{(x_i, H')\}\sigma) \geq e-k$, and similarly $\text{EqLv}(F'_j\sigma, F_{s+j}\sigma) \geq e-k$, by using Prop. 5 we get

$$\text{EqLv}(E'_j\sigma, F'_j\sigma) = \text{EqLv}(E_{s+j}\sigma, F_{s+j}\sigma).$$

Since $(E'_j\sigma, F'_j\sigma) = (E'_j\sigma_{[-x_i]}, F'_j\sigma_{[-x_i]})$, we get

$$e-k > \text{EqLv}(E'_1\sigma_{[-x_i]}, F'_1\sigma_{[-x_i]}) > \dots > \text{EqLv}(E'_{\ell-s}\sigma_{[-x_i]}, F'_{\ell-s}\sigma_{[-x_i]}).$$

Finally we note that

$$\begin{aligned} \text{PrSize}(E'_j, F'_j) &\leq \text{PrSize}(E_{j+s}, F_{j+s}) + 2 \cdot \text{PrSize}(H) \leq \\ &\leq g(j+s) + 2 \cdot (g(1) + \text{MEL}_{g(1)} \cdot \text{STEPINC}) = g'(j), \end{aligned}$$

for g' defined by (9). Hence

$$(E'_1\sigma_{[-x_i]}, F'_1\sigma_{[-x_i]}), (E'_2\sigma_{[-x_i]}, F'_2\sigma_{[-x_i]}), \dots, (E'_{\ell-s}\sigma_{[-x_i]}, F'_{\ell-s}\sigma_{[-x_i]})$$

is an eqlevel-decreasing $(n-1, g')$ -sequence. By the induction hypothesis we have $\ell-s \leq \ell_{n-1, g'}$, and thus $\ell \leq 1 + \text{MEL}_{g(1)} + \ell_{n-1, g'} = \ell_{n, g}$. \square

If $T_0 \sim U_0$, then Prover can force a potentially infinite (n_0, g_0) -sequence for certain n_0, g_0 determined by \mathcal{G} (by Lemma 7). She could claim a win after creating an (n_0, g_0) -sequence longer than ℓ_{n_0, g_0} , if she could demonstrate the value ℓ_{n_0, g_0} . (By Lemma 10 it would be then clear that Refuter does not use the least-eqlevel strategy or $T_0 \sim U_0$.) Inspecting the above, we can verify that for computing $\ell_{n, g}$ for concrete n, g it suffices to know $\text{SIZE}_{\leq B} \cap \not\sim$ for a sufficiently large $B \in \mathbb{N}$, and the values $g(j)$ for j from a sufficiently large initial segment $[1, M]$ of \mathbb{N} . We can capture this by the following inductive definition:

- We say that $B \in \mathbb{N}$ is a *sufficient size-bound* for $n \in \mathbb{N}$ and $g : \mathbb{N}_+ \rightarrow \mathbb{N}_+$ (i.e., for computing $\ell_{n, g}$) if $g(1) \leq B$ and in the case $n > 0$ we also have that B is a sufficient size-bound for $n-1, g'$ where g' is defined by (9).
- We say that $M \in \mathbb{N}$ is a *sufficient segment-bound* for $n \in \mathbb{N}$ and $g : \mathbb{N}_+ \rightarrow \mathbb{N}_+$ (i.e., for computing $\ell_{n, g}$) if $M \geq 1$ and in the case $n > 0$ we have that $M \geq 1 + \text{MEL}_{g(1)} + M'$ where M' is a sufficient segment-bound for $n-1, g'$ where g' is defined by (9).

Finally we note that Prover can, when given a grammar \mathcal{G} , present some $n_0 \in \mathbb{N}$ and $g_0 : \mathbb{N}_+ \rightarrow \mathbb{N}_+$ (or just the values $g_0(1), g_0(2), \dots, g_0(M)$ for some $M \in \mathbb{N}$) and perform the above recursive computation for ℓ_{n_0, g_0} , while guessing a set $\mathcal{C} \subseteq (\text{SIZE}_{\leq B} \cap \not\sim)$ for some $B \in \mathbb{N}$ that is sufficient for this computation. We note that Prover can demonstrate that $\mathcal{C} \subseteq \not\sim$, and also compute the eq-level for each pair in \mathcal{C} (recall Prop. 2). For $\text{REST} = \text{SIZE}_{\leq B} \setminus \mathcal{C}$ Prover just claims that it is a subset of \sim , in which case her computation of ℓ_{n_0, g_0} would be indeed correct; in reality she computes a value $\ell_{n_0, g_0}^{\mathcal{C}}$ that is dependent on her choice of \mathcal{C} . We let Refuter to challenge the assumption $\text{REST} \subseteq \sim$, by choosing a pair from REST , so that his least-eqlevel strategy will still be winning if Prover does not guess \mathcal{C} correctly. This idea will be now formalized and embodied in the final game-version.

For $\mathcal{C} \subseteq \not\sim$ and $b \in \mathbb{N}$ we put

$$\text{MEL}_b^{\mathcal{C}} = \text{MAXEQL}(\text{SIZE}_{\leq b} \cap \mathcal{C}).$$

For triples (\mathcal{C}, n, g) where $\mathcal{C} \subseteq \mathcal{A}$, $n \in \mathbb{N}$, $g : \mathbb{N}_+ \rightarrow \mathbb{N}_+$ we define $\ell_{n,g}^{\mathcal{C}}$ by the following recursive definition:

$$\begin{aligned} \ell_{0,g}^{\mathcal{C}} &= 1 + \text{MEL}_{g(1)}^{\mathcal{C}}, \text{ and} \\ \ell_{n+1,g}^{\mathcal{C}} &= 1 + \text{MEL}_{g(1)}^{\mathcal{C}} + \ell_{n,g'} \text{ where} \end{aligned}$$

$$g'(j) = g(1 + \text{MEL}_{g(1)}^{\mathcal{C}} + j) + 2 \cdot (g(1) + \text{MEL}_{g(1)}^{\mathcal{C}} \cdot \text{STEPINC}) \text{ for all } j \in \mathbb{N}_+. \quad (10)$$

- We say that $B \in \mathbb{N}$ is a *sufficient size-bound* for a triple (\mathcal{C}, n, g) as above if $\mathcal{C} \subseteq \text{SIZE}_{\leq B}$, $g(1) \leq B$, and in the case $n > 0$ we also have that B is a sufficient size-bound for $(\mathcal{C}, n-1, g')$ where g' is defined by (10).
- We say that $M \in \mathbb{N}$ is a *sufficient segment-bound* for (\mathcal{C}, n, g) if $M \geq 1$, and in the case $n > 0$ we have that $M \geq 1 + \text{MEL}_{g(1)}^{\mathcal{C}} + M'$ where M' is a sufficient segment-bound for $(\mathcal{C}, n-1, g')$ where g' is defined by (10).

We now derive an analogy of Lemma 10:

Lemma 11. *Let B be a sufficient size-bound for (\mathcal{C}, n, g) , where $\mathcal{C} \subseteq \mathcal{A}$, $n \in \mathbb{N}$, $g : \mathbb{N}_+ \rightarrow \mathbb{N}_+$, and let $\text{REST} = \text{SIZE}_B \setminus \mathcal{C}$. Then any eqlevel-decreasing (n, g) -sequence starting with a pair whose eq-level is less than $\text{MINEQL}(\text{REST})$ has length at most $\ell_{n,g}^{\mathcal{C}}$.*

Proof. Let the assumption hold, and let us have an (n, g) -sequence

$$(E_1\sigma, F_1\sigma), (E_2\sigma, F_2\sigma), \dots, (E_\ell\sigma, F_\ell\sigma) \text{ where} \\ \text{EQLV}(E_1\sigma, F_1\sigma) < \text{MINEQL}(\text{REST}).$$

Since $\text{EQLV}(E_1, F_1) \leq \text{EQLV}(E_1\sigma, F_1\sigma)$, and $\text{PRSIZE}(E_1, F_1) \leq g(1) \leq B$ (and thus $(E_1, F_1) \in \text{SIZE}_{\leq B}$ and $\text{EQLV}(E_1, F_1) < \text{MINEQL}(\text{REST})$, which entails $(E_1, F_1) \notin \text{REST}$), we must have $(E_1, F_1) \in \mathcal{C}$; therefore $\text{EQLV}(E_1, F_1) \leq \text{MEL}_{g(1)}^{\mathcal{C}}$.

If $n = 0$, or more generally if $\text{EQLV}(E_1, F_1) = \text{EQLV}(E_1\sigma, F_1\sigma)$, then

$$\ell \leq 1 + \text{EQLV}(E_1, F_1) \leq 1 + \text{MEL}_{g(1)}^{\mathcal{C}} = \ell_{0,g}^{\mathcal{C}}.$$

If $(n > 0 \text{ and}) \text{EQLV}(E_1, F_1) = k < e = \text{EQLV}(E_1\sigma, F_1\sigma)$, then

$$\sigma \sim_{e-k} \{(x_i, H')\}\sigma_{[-x_i]}$$

for some $x_i \in \text{SUPP}(\sigma)$ and some H' with $\text{PRSIZE}(H') \leq g(1) + k \cdot \text{STEPINC} \leq g(1) + \text{MEL}_{g(1)}^{\mathcal{C}} \cdot \text{STEPINC}$ (by Prop. 8 and 9).

We now put (*shift*) $s = 1 + \text{MEL}_{g(1)}^{\mathcal{C}}$; hence $s > \text{EQLV}(E_1, F_1) = k$, and thus

$$e - k > \text{EQLV}(E_{s+1}\sigma, F_{s+1}\sigma) > \dots > \text{EQLV}(E_{s+(\ell-s)}\sigma, F_{s+(\ell-s)}\sigma).$$

For $j = 1, 2, \dots, \ell - s$ we define $(E'_j, F'_j) = (E_{s+j}\{(x_i, H')\}, F_{s+j}\{(x_i, H')\})$; we thus have $\text{PRSIZE}(E'_j, F'_j) \leq g'(j)$ for g' defined by (10). We have $(E'_j\sigma, F'_j\sigma) = (E'_j\sigma_{[-x_i]}, F'_j\sigma_{[-x_i]})$, and by using Prop. 5 we also derive

$$\text{EqLv}(E'_j\sigma_{[-x_i]}, F'_j\sigma_{[-x_i]}) = \text{EqLv}(E_{s+j}\sigma, F_{s+j}\sigma).$$

Hence the sequence

$$(E'_1\sigma_{[-x_i]}, F'_1\sigma_{[-x_i]}), (E'_2\sigma_{[-x_i]}, F'_2\sigma_{[-x_i]}), \dots, (E'_{\ell-s}\sigma_{[-x_i]}, F'_{\ell-s}\sigma_{[-x_i]})$$

is an eqlevel-decreasing $(n-1, g')$ -sequence, and B is sufficient for $(\mathcal{C}, n-1, g')$ (since B is assumed sufficient for (\mathcal{C}, n, g)). The induction hypothesis thus implies that $\ell-s \leq \ell_{n-1, g'}^{\mathcal{C}}$, and thus $\ell \leq 1 + \text{MEL}_{g(1)}^{\mathcal{C}} + \ell_{n-1, g'}^{\mathcal{C}} = \ell_{n, g}^{\mathcal{C}}$. \square

Prover-Refuter game (third version). We separate \mathcal{G} from the initial pair, now denoted (E_0, F_0) , to stress that the initial phase depends on \mathcal{G} only.

- i) A grammar $\mathcal{G} = (\mathcal{N}, \Sigma, \mathcal{R})$ is given.
- ii) Prover provides some finite set $\mathcal{C} \subseteq \text{TERMS}_{\mathcal{N}} \times \text{TERMS}_{\mathcal{N}}$, some $n_0 \in \mathbb{N}$, a sequence of increasing values denoted $g_0(1), g_0(2), \dots, g_0(M)$ for some $M \in \mathbb{N}$, and some $B \in \mathbb{N}$ such that $\mathcal{C} \subseteq \text{SIZE}_{\leq B}$. For each pair $(E, F) \in \mathcal{C}$ Prover provides $e \in \mathbb{N}$ and demonstrates that $\text{EqLv}(E, F) = e$ (recall Prop. 2); thus $\mathcal{C} \subseteq \mathcal{N}$. Prover now computes $\ell_{n_0, g_0}^{\mathcal{C}}$, using the recursive definition given before (10); this fails when B or M are not sufficiently large.
- iii) An initial pair (E_0, F_0) is given.
- iv) For $\text{REST} = \text{SIZE}_{\leq B} \setminus \mathcal{C}$, Refuter chooses (T_0, U_0) from $\{(E_0, F_0)\} \cup \text{REST}$ (with the least eq-level when using the least-eqlevel strategy).
- v) Now a play of the second game-version starts with (T_0, U_0) . A new feature is that Prover can claim her win when she shows that $(T_1, U_1), (T_2, U_2), \dots$ contains an (n, g) -subsequence that is longer than $\ell_{n_0, g_0}^{\mathcal{C}}$.

The least-eqlevel strategy still guarantees Refuter's win for $E_0 \not\sim F_0$; Prover can never win by the new game-rule (i.e., by exceeding $\ell_{n_0, g_0}^{\mathcal{C}}$), due to Lemma 11. On the other hand, Prover can correctly guess $\mathcal{C} = \text{SIZE}_{\leq B} \cap \mathcal{N}$ for B that is sufficient for computing (the real) ℓ_{n_0, g_0} (related to n_0, g_0 that are guaranteed for \mathcal{G} by Lemma 7), and she can force her win when $E_0 \sim F_0$.

Since a winning strategy of Prover (for any \mathcal{G}, E_0, F_0 where $E_0 \sim F_0$) is finitely presentable and effectively verifiable (which easily follows from the fact that Refuter always has only finitely many options when it is his turn), a proof of Theorem 1 is now clear.

4 Additional Remarks

Theorem 1 just states the existence of an algorithm deciding *bisimulation equivalence* of first-order grammars, or, in more detail, computing the respective eq-levels. But the proof can be surely adapted to more general statements. It would be a technical exercise to phrase the proof in some more general terms, not referring to bisimilarity. E.g., we could speak about some more general (stratified) equivalence with a related notion of covering $\mathcal{B} \triangleleft \mathcal{B}'$ with some properties like those captured in Prop. 3, etc. As usual, a question in such cases is to what extent it makes good sense. E.g., do we get new worthwhile decidability results in such a way?

If we look at a (straightforward) transformation from pushdown automata (PDA) to FO-grammars (given here in Appendix for completeness), we note how FO-grammars “swallow” deterministic popping ε -steps in PDA. (If there is no other rule for A than $A(x_1, \dots, x_m) \xrightarrow{\varepsilon} x_i$, then any (sub)term $A(G_1, \dots, G_m)$ can be immediately replaced with G_i .) A question posed by Stirling was if bisimilarity of PDA with just popping ε -steps (where some nondeterminism is allowed) is still decidable. This was answered negatively in [9].

In our term-framework we extend the action set in $\mathcal{G} = (\mathcal{N}, \Sigma, \mathcal{R})$ with a *silent action*, denoted ε , and we also allow ε -rules $A(x_1, \dots, x_m) \xrightarrow{\varepsilon} E$. The associated LTS is then $\mathcal{L}_{\mathcal{G}}^A = (\text{TERMS}_{\mathcal{N}}, \Sigma \cup \{\varepsilon\}, (\xrightarrow{a})_{a \in \Sigma \cup \{\varepsilon\}})$ that naturally extends the LTS $\mathcal{L}_{\mathcal{G}}^A$ defined for the case with no ε -rules. The *collapsed LTS* $\mathcal{L}_{\mathcal{G}}^{A-\text{COL}}$ arises from $\mathcal{L}_{\mathcal{G}}^A$ by “swallowing” the ε -transitions, i.e., we have no ε -transitions in $\mathcal{L}_{\mathcal{G}}^{A-\text{COL}}$, and $F \xrightarrow{a} H$ in $\mathcal{L}_{\mathcal{G}}^{A-\text{COL}}$ if in $\mathcal{L}_{\mathcal{G}}^A$ there is a path of the form

$$F = F_0 \xrightarrow{\varepsilon} F_1 \xrightarrow{\varepsilon} \dots \xrightarrow{\varepsilon} F_{k_1} \xrightarrow{a} H_0 \xrightarrow{\varepsilon} H_1 \xrightarrow{\varepsilon} \dots \xrightarrow{\varepsilon} H_{k_2} = H. \quad (11)$$

A construction in [9] shows that bisimilarity in $\mathcal{L}_{\mathcal{G}}^{A-\text{COL}}$ is undecidable, even when all ε -rules are popping, i.e. of the form $A(x_1, \dots, x_m) \xrightarrow{\varepsilon} x_i$. As also noted in [9], the same proof construction also shows that *weak bisimilarity* (for PDA or for $\mathcal{L}_{\mathcal{G}}^A$ of FO-grammars where silent popping moves are allowed) is undecidable.

In fact, the construction for undecidability in [9] works also when we do not include the silent “post-transitions”, i.e., if we require $k_2 = 0$ in (11); thus the undecidability also holds for the respective equivalence that is finer than weak bisimilarity.

The undecidability results have been recently refined, using branching bisimilarity [22].

In branching bisimilarity we also exclude the silent “post-transitions” (as mentioned above) but there is also a “semantical” constraint: the silent “pre-transitions” are supposed to be not changing the equivalence-class. Formally, given $\mathcal{G} = (\mathcal{N}, \Sigma, \mathcal{R})$, where rules $A(x_1, \dots, x_m) \xrightarrow{\varepsilon} G$ are allowed, we can define a *branching bisimulation* in the (non-collapsed) LTS $\mathcal{L}_{\mathcal{G}}^A$ as a symmetric relation $\mathcal{B} \subseteq \text{TERMS}_{\mathcal{N}} \times \text{TERMS}_{\mathcal{N}}$ where for each move $E \xrightarrow{a} E'$ in a pair $(E, F) \in \mathcal{B}$, for $a \in \Sigma \cup \{\varepsilon\}$, there is a sequence, a *response*, $F = F_0 \xrightarrow{\varepsilon} F_1 \xrightarrow{\varepsilon} \dots \xrightarrow{\varepsilon} F_k \xrightarrow{a} F'$ such that $(E, F_i) \in \mathcal{B}$ for all $i \in [0, k]$, and $(E', F') \in \mathcal{B}$; if $a = \varepsilon$, then it suffices that $(E', F) \in \mathcal{B}$ (i.e., the response might be empty).

For “pushing” ε -rules the construction for undecidability from [9] can be again easily adapted to branching bisimilarity. But if we only allow popping ε -rules, of the form $A(x_1, \dots, x_m) \xrightarrow{\varepsilon} x_i$, then the construction from [9] cannot be used for branching bisimilarity; there the silent “pre-transitions” do not keep the same equivalence class. This was noted by Y. Fu and Q. Yin [7] who announced a result that would translate in our setting as the *decidability of branching bisimilarity of FO-grammars with popping ε -rules*.

In fact, [7] also announces the decidability for pushing ε -rules in the context of so called normed PDA processes. The crucial idea is that the above “responses” $F = F_0 \xrightarrow{\varepsilon} F_1 \xrightarrow{\varepsilon} \dots \xrightarrow{\varepsilon} F_k \xrightarrow{a} F'$ to the moves $E \xrightarrow{a} E'$ can be bounded in this context; i.e., Prover gets again only boundedly many possibilities how to cover a given finite set \mathcal{B} (in the adapted version of $\mathcal{B} \triangleleft \mathcal{B}'$), which allows us to proceed essentially in the way that we used in this paper.

Hence Y. Fu and Q. Yin have noticed that it indeed makes good sense to try to adapt the decidability proof for bisimilarity to get further results. An adaptation of an existing proof seems necessary, since the branching bisimilarity problem that they study, in particular for PDA with popping ε -steps, does not seem to be easily reducible to the known decidable (bisimilarity) problem; one has thus to delve into the existing proofs, looking for their possible adaptations.

Y. Fu and Q. Yin have chosen to build on Stirling’s paper [21]. They adapt Stirling’s tableau approach to a new model that they invented. In fact, when one looks at their technical model, it seems clear that it could be smoothly replaced with the first-order-term framework used here (and already in [10], i.e., in the paper of which the authors of [7] became aware only afterwards, as they say in their conclusions). Analysing their procedure and its relation to our Prover-Refuter game would require a nontrivial technical work; here we thus suggest a direct adaptation of the game that captures the announced result.

Adaptation of Prover-Refuter game. If we want to use our framework of the Prover-Refuter game directly to branching bisimilarity of FO-grammars with popping ε -rules, thus modifying the relation $\mathcal{B} \triangleleft \mathcal{B}'$ accordingly, we encounter a technical problem. Though in a pair (E, F) each move $E \xrightarrow{a} E'$ ($a \in \Sigma \cup \{\varepsilon\}$) still has only finitely many possible responses $F = F_0 \xrightarrow{\varepsilon} F_1 \xrightarrow{\varepsilon} \dots \xrightarrow{\varepsilon} F_k \xrightarrow{a} F'$, their number is not bounded (by a quantity determined by the underlying grammar \mathcal{G}), since the responses might be sinking to subterms of F in unbounded depths. This causes, e.g., that the bal-result related to a pivot might not be “boundedly close” to the pivot. But we can require that Prover avoids such unbounded responses; she can always tell, whenever she presents a new (sub)term V , if V is equivalent with a root-successor V' in V , and she must then behave consistently with her claims; we can imagine that she colours the respective arcs (from the root of V to the root of V') as “blue”.

Recall that any response $F = F_0 \xrightarrow{\varepsilon} F_1 \xrightarrow{\varepsilon} \dots \xrightarrow{\varepsilon} F_k \xrightarrow{a} F'$ should not change the equivalence class when “sinking” from F to F_k ; this sinking can be done only along such blue arcs if Prover colours the arcs correctly.

Such a blue arc, i.e. a claim that $V \sim V'$ where V' is a root-successor in V , can be also challenged by Refuter, but it can be used (later) for replacing V with V' when this should not affect the current eq-level, if Refuter uses the least-eql-level strategy. In this way the respective possible transitions $V \xrightarrow{\varepsilon} V'$ (where $V \sim V'$) are also “swallowed”, similarly as deterministic popping ε -steps, which are swallowed “automatically”.

Recall that if there is no other rule for A than $A(x_1, \dots, x_m) \xrightarrow{\varepsilon} x_i$, then any (sub)term $A(G_1, \dots, G_m)$ can be immediately replaced with G_i .

We thus recover the “bounded-closeness” properties, and we can accordingly adapt the proof that was used in the case with no ε -steps.

Below we suggest a possible way how to formalize the above idea of “blue arcs”. We stay in the framework of bisimilarity of FO-grammars (with no ε -rules), but the decidability for branching bisimilarity (of FO-grammars with popping ε -steps) follows routinely after this adaptation. We thus consider a grammar $\mathcal{G} = (\mathcal{N}, \Sigma, \mathcal{R})$ and the Prover-Refuter game as they were defined previously.

Quotient graph-presentations, and related decompositions.

We now stress more explicitly that we deal with finite objects in the Prover-Refuter game, i.e., with graph-presentations (of terms), called just *graphs* in what follows, rather than with the terms themselves. We use symbols E, F, \dots, T, U, \dots to range over graphs (while E, F, \dots, T, U, \dots range over terms).

In fact, we can take graphs as the states in the LTSs $\mathcal{L}_{\mathcal{G}}^R, \mathcal{L}_{\mathcal{G}}^A$; we made clear how the transitions look like in this case. Nevertheless, our aim is to convey the main idea, not to delve into unnecessary technicalities.

Each graph V has finitely many nodes, and each node \mathbf{n} corresponds to the term $\text{TERM}(\mathbf{n})$ rooted in \mathbf{n} .

We write shortly $\mathbf{n}_1 \sim_k \mathbf{n}_2$ instead of $\text{TERM}(\mathbf{n}_1) \sim_k \text{TERM}(\mathbf{n}_2)$; similarly we write $\text{EQLV}(\mathbf{n}_1, \mathbf{n}_2)$ instead of $\text{EQLV}(\text{TERM}(\mathbf{n}_1), \text{TERM}(\mathbf{n}_2))$.

By $\text{AREA}(\mathbf{n})$ for a node \mathbf{n} of V we mean the restriction of V to the nodes occurring on (directed) paths in V that start in \mathbf{n} .

We thus have a correspondence (not necessarily one-to-one) between the nodes in $\text{AREA}(\mathbf{n})$ and the subterms of $\text{TERM}(\mathbf{n})$.

Let us now consider a graph V with a partition \mathcal{P} of its nodes. By $V_{[\mathcal{P}]}$ we denote a (chosen) *quotient of V w.r.t. \mathcal{P}* arising as follows: In each partition-class of \mathcal{P} we choose a representant-node; the nodes of $V_{[\mathcal{P}]}$ are the representant-nodes of all partition classes, and if an arc from a representant leads to a non-representant \mathbf{n} , then we redirect it to the representant \mathbf{n}' of the partition-class containing \mathbf{n} . We thus also get a mapping $\text{RED}_{\mathcal{P}}$, “*reducing*” each node \mathbf{n} of V to $\text{RED}_{\mathcal{P}}(\mathbf{n})$, which is the node in $V_{[\mathcal{P}]}$ representing the partition-class of \mathbf{n} .

For a node \mathbf{n} of V , by $\text{D1-RED}_{\mathcal{P}}(\mathbf{n})$ (“depth-1-reducing of \mathbf{n} ”) we mean (a copy of) the node \mathbf{n} in the graph arising as follows: we take a disjoint union of V and $V_{[\mathcal{P}]}$, where each outgoing arc of \mathbf{n} in V , leading to some \mathbf{n}' , is redirected to the node $\text{RED}_{\mathcal{P}}(\mathbf{n}')$ in $V_{[\mathcal{P}]}$. Thus the nodes in $\text{AREA}(\text{D1-RED}_{\mathcal{P}}(\mathbf{n}))$ are taken from $V_{[\mathcal{P}]}$, with the exception of the root.

We now define the *decomposition of V by \mathcal{P}* :

$$\text{DEC}_{\mathcal{P}}(V) = \{(\text{D1-RED}_{\mathcal{P}}(\mathbf{n}), \text{RED}_{\mathcal{P}}(\mathbf{n})) \mid \mathbf{n} \text{ is a node in } V\}.$$

Hence $\text{DEC}_{\mathcal{P}}(V)$ is a set of pairs of nodes in a graph; the graph arises from $V_{[\mathcal{P}]}$ by adding copies of the nodes from V whose outgoing arcs are directed into $V_{[\mathcal{P}]}$.

Proposition 12. *Let V be a graph and \mathcal{P} a partition of its nodes. If $\mathbf{n}_1, \mathbf{n}_2$ are in the same partition-class of \mathcal{P} , then $\text{EQLV}(\mathbf{n}_1, \mathbf{n}_2) \geq \text{MINEQL}(\text{DEC}_{\mathcal{P}}(V))$.*

Proof. Suppose some V, \mathcal{P} ; let $\text{MINEQL}(\text{DEC}_{\mathcal{P}}(V)) = e$. By the congruence properties we derive for each node \mathbf{n} in V that

$$\text{EQLV}(\mathbf{n}, \text{RED}_{\mathcal{P}}(\mathbf{n})) \geq e \text{ and } \text{EQLV}(\mathbf{n}, \text{D1-RED}_{\mathcal{P}}(\mathbf{n})) \geq e.$$

We can show this as follows. Let $e' = \text{MINEQL}(\{(\mathbf{n}, \text{RED}_{\mathcal{P}}(\mathbf{n})) \mid \mathbf{n} \text{ is a node in } V\})$. Then $\mathbf{n} \sim_{e'+1} \text{D1-RED}_{\mathcal{P}}(\mathbf{n})$, by Prop. 6(2). We thus have $\mathbf{n} \sim_{e'} \text{RED}_{\mathcal{P}}(\mathbf{n}) \sim_e \text{D1-RED}_{\mathcal{P}}(\mathbf{n}) \sim_{e'+1} \mathbf{n}$, which implies $e' \geq e$.

For $\mathbf{n}_1, \mathbf{n}_2$ where $\text{RED}_{\mathcal{P}}(\mathbf{n}_1) = \text{RED}_{\mathcal{P}}(\mathbf{n}_2)$ we thus have

$$\mathbf{n}_1 \sim_e \text{D1-RED}_{\mathcal{P}}(\mathbf{n}_1) \sim_e \text{RED}_{\mathcal{P}}(\mathbf{n}_1) = \text{RED}_{\mathcal{P}}(\mathbf{n}_2) \sim_e \mathbf{n}_2.$$

□

We will particularly use the decompositions of graphs V that are induced by sets of arcs in V (later called “blue arcs” or “red arcs”). Suppose V and a set of its arcs; we call the arcs in the set “blue”. This defines the least partition where the source-node and the target-node of any blue arc are in the same partition-class.

A partition \mathcal{P} of the nodes of V determines the set $\{(\mathbf{n}_1, \mathbf{n}_2) \mid \mathbf{n}_1, \mathbf{n}_2 \text{ are in the same partition-class of } \mathcal{P}\}$. Hence partitions can be naturally ordered by inclusion; we refer to this order when saying “the least partition such that ...”. In the above case, two nodes are in the same partition-class iff there is a “blue-path” between them in the *undirected* graph version.

Modified Prover-Refuter game. Let us recall the second version of the game. We modify it as follows.

1. We denote the initial pair $(\overline{T}, \overline{U})$, and assume that it is given by a graph V with two designated nodes $\mathbf{n}', \mathbf{n}''$ where $\text{TERM}(\mathbf{n}') = \overline{T}$ and $\text{TERM}(\mathbf{n}'') = \overline{U}$. Prover now suggests a partition \mathcal{P} of the set of nodes of V (generally, not necessarily by “blue arcs”) where $\mathbf{n}', \mathbf{n}''$ must be in the same partition-class. Now $V_{[\mathcal{P}]}$ is created, where each non-loop arc is coloured *black*; in this way Prover claims that $\mathbf{n}_1 \not\sim \mathbf{n}_2$ for the source-node \mathbf{n}_1 and the target-node \mathbf{n}_2 .

Any loop-arc from \mathbf{n} to \mathbf{n} trivially satisfies that its source-node and its target-node are equivalent; we further ignore such arcs in our discussion.

We also note that Prover could even demonstrate that $\mathbf{n}_1 \not\sim \mathbf{n}_2$ but this is not necessary here.

Prover claims that the above partition \mathcal{P} is induced by the bisimulation equivalence; she thus also claims that $\text{MINEQL}(\text{DEC}_{\mathcal{P}}(V)) = \omega$.

Refuter now chooses a pair $(\text{D1-RED}_{\mathcal{P}}(\mathbf{n}), \text{RED}_{\mathcal{P}}(\mathbf{n}))$ from $\text{DEC}_{\mathcal{P}}(V)$, corresponding to a pair (T_0, U_0) of terms.

If Refuter uses the least-eqllevel strategy, we have $\text{EqLv}(T_0, U_0) \leq \text{EqLv}(\overline{T}, \overline{U})$ (by Prop. 12).

The pair (T_0, U_0) is thus, in fact, given by a pair $(\mathbf{n}_{01}, \mathbf{n}_{02})$ of nodes of a graph V_0 where only the outgoing arcs of \mathbf{n}_{01} might be not black (when ignoring the loop-arcs). Prover is supposed to colour each outgoing arc of \mathbf{n}_{01} as “blue” iff its source-node and its target-node are bisimilar.

2. Prover will use a strategy (corresponding to the strategy in the proof of Lemma 7) that also guarantees that the $(i+1)$ -th round starts with a pair (T_i, U_i) given by two nodes $\mathbf{n}_{i1}, \mathbf{n}_{i2}$ in some graph V_i where the arcs are coloured black or blue (ignoring the loop-arcs), and where a cycle in V_i never contains a blue arc, and each blue arc is in a bounded distance (depth) from \mathbf{n}_{i1} or \mathbf{n}_{i2} .

As previously, by a “bounded” depth we mean that the respective bound is determined by the underlying grammar \mathcal{G} .

3. Whenever Prover presents a new graph (in the sets \mathcal{B}_j), she is supposed to colour each arc, from \mathbf{n} to \mathbf{n}' , *blue* if $\mathbf{n} \sim \mathbf{n}'$; by black arcs she claims non-equivalence. She must be consistent with her previous choices.

Now the sets \mathcal{B}_j contain graphs with two designated nodes, and with coloured arcs. Refuter thus chooses (T'_i, U'_i) by choosing a graph V'_i with two designated nodes $\mathbf{n}'_{i1}, \mathbf{n}'_{i2}$.

4. If Prover does not make a balancing step, in the $(i+1)$ -th round after Refuter has chosen V'_i with $\mathbf{n}'_{i1}, \mathbf{n}'_{i2}$, then we define the partition \mathcal{P} of the nodes in V'_i as the least partition containing $(\mathbf{n}'_{i1}, \mathbf{n}'_{i2})$ and the source-target pairs of all blue arcs. Refuter chooses a pair $(\text{D1-RED}_{\mathcal{P}}(\mathbf{n}), \text{RED}_{\mathcal{P}}(\mathbf{n}))$ from $\text{DEC}_{\mathcal{P}}(V'_i)$, which presents the pair (T_{i+1}, U_{i+1}) .
5. Suppose that Prover makes a balancing step, say a left one, corresponding to replacing $(T'_i, U'_i) = (G\sigma, U'_i)$ with $(G\sigma', U'_i)$; this is naturally implemented in the graph, and we get a graph V''_i instead of V'_i . In V''_i we recolour the blue arcs in $\text{AREA}(\mathbf{n}'_{i2})$ (in the U'_i -area) to *red*; we perform such a blue-to-red recolouring also in the “ $\text{RANGE}(\sigma')$ -area”, i.e., in $\text{AREA}(\mathbf{n})$ for each \mathbf{n} that corresponds to the root of some $V' \in \text{RANGE}(\sigma')$. Now we define the partition \mathcal{P} of the nodes in V''_i as the least partition containing $(\mathbf{n}'_{i1}, \mathbf{n}'_{i2})$ and the source-target pairs of all *red* arcs. Refuter chooses a pair $(\text{D1-RED}_{\mathcal{P}}(\mathbf{n}), \text{RED}_{\mathcal{P}}(\mathbf{n}))$ from $\text{DEC}_{\mathcal{P}}(V''_i)$, which presents the pair (T_{i+1}, U_{i+1}) .

In the respective graph V_{i+1} we have no red arcs but there can be the blue arcs inherited from the special head G , which has a bounded height.

When Prover uses the strategy described in the proof of Lemma 7, while also guessing the blue arcs correctly, she cannot lose when $\overline{T} \sim \overline{U}$. It is a routine to verify that any bal-result is still “boundedly-close” to its respective pivot. Now the pivots W, W' of two consecutive balancings (not necessarily in two consecutive rounds) might not satisfy that W' is boundedly reachable from a subterm of W , but W' arises from a term boundedly reachable from a subterm of W by some replacings of bounded-depth subterms with other bounded-depth subterms. This fact enables to derive Lemma 7 as previously.

In the case with no ε -rules, the “machinery” of blue arcs is not needed. But it makes sense when we consider branching bisimilarity in the case of popping ε -rules. There the relation $\mathcal{B} \triangleleft \mathcal{B}'$ is modified appropriately, and we allow Prover to use only responses $F = F_0 \xrightarrow{\varepsilon} F_1 \xrightarrow{\varepsilon} \dots \xrightarrow{\varepsilon} F_k \xrightarrow{a} F'$ where the “pre-transitions” can only sink along blue arcs, and thus always into bounded depths. The decidability proof can be then finished analogously to the case with no ε -rules.

Further remarks on related research. Further work is needed to fully understand the discussed problems. E.g., even the case of BPA processes, generated by real-time PDA with a single control-state, is not quite clear. Here the bisimilarity problem is EXPTIME-hard [13] and in 2-EXPTIME [3] (proven explicitly in [11]); for the subclass of normed BPA the problem is polynomial [8] (see [6] for the best published upper bound).

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Appendix

A transformation of PDA to first-order grammars.

By a *pushdown automaton* (PDA) we mean a structure $\mathcal{M} = (Q, \Gamma, \Sigma, \Delta)$ where Q, Γ, Σ are finite sets of *control states*, of *stack symbols*, and of *actions* (or *input letters*), respectively; Δ is a finite set of *pushdown-rules* of the form $pX \xrightarrow{a} q\alpha$ where $p, q \in Q$, $X \in \Gamma$, $\alpha \in \Gamma^*$, and $a \in \Sigma \cup \{\varepsilon\}$. By a *configuration* we mean any string $q\beta\perp$ where $q \in Q$, $\beta \in \Gamma^*$, and \perp is a special *bottom-of-the-stack symbol* (where $\perp \notin \Gamma$).

A PDA $\mathcal{M} = (Q, \Gamma, \Sigma, \Delta)$ has the associated LTS

$$\mathcal{L}_{\mathcal{M}} = (\text{CONF}, \Sigma \cup \{\varepsilon\}, (\xrightarrow{a})_{a \in \Sigma \cup \{\varepsilon\}})$$

where CONF is the set of configurations, and the transitions are induced by the pushdown-rules as follows:

$$\text{if } pX \xrightarrow{a} q\alpha \text{ is in } \Delta, \text{ then } pX\beta\perp \xrightarrow{a} q\alpha\beta\perp \text{ for any } \beta \in \Gamma^*.$$

Suppose $Q = \{q_1, q_2, \dots, q_m\}$. Then a configuration $q_i Y_1 Y_2 \dots Y_k \perp$ can be naturally viewed as the term $\mathcal{T}(q_i Y_1 Y_2 \dots Y_k \perp)$ defined inductively by the points 1 and 2 below. Hence we view each pair (q_i, Y) of a control state and a stack symbol as a nonterminal $[q_i Y]$ with arity m ; a special case is \perp with arity 0. A pushdown rule $q_i Y \xrightarrow{a} q_j \beta$ is rewritten to $q_i Y x \xrightarrow{a} q_j \beta x$ for a special formal symbol x , and

$$q_i Y x \xrightarrow{a} q_j \beta x \text{ is transformed to } \mathcal{T}(q_i Y x) \xrightarrow{a} \mathcal{T}(q_j \beta x),$$

where we also use the point 3 below:

1. $\mathcal{T}(q_i \perp) = \perp$,
2. $\mathcal{T}(q_i Y \alpha) = [q_i Y](\mathcal{T}(q_1 \alpha), \mathcal{T}(q_2 \alpha), \dots, \mathcal{T}(q_m \alpha))$.

3. $\mathcal{T}(q_j x) = x_j$.

Hence $\mathcal{T}(q_i Y x) = [q_i Y](x_1, x_2, \dots, x_m)$. In fact, we can modify the operator \mathcal{T} for *deterministic popping ε -rules*: If there is no other pushdown-rule for q_i, Y than $q_i Y \xrightarrow{\varepsilon} q_j$, then instead of creating the grammar rule $\mathcal{T}(q_i Y x) \xrightarrow{\varepsilon} \mathcal{T}(q_j x)$ we might modify the transformation \mathcal{T} by putting $\mathcal{T}(q_i Y \alpha) = \mathcal{T}(q_j \alpha)$; we have thus “*swallowed*” the respective ε -step. (The branches of the syntactic tree of $\mathcal{T}(q \alpha)$ can have varying lengths in this case.)

We thus do not need ε -rules in FO-grammars for expressing PDA where only deterministic popping ε -moves are allowed.